

LOS MÉTODOS DE INTEGRACION

FOR

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(Continuacion)

$$8. \int_x \pi (m_0 x + c x^{s+1}) a^2 x^{2n} = a^2 \pi \int (m_0 x^{2n+1} + c x^{s+2n+1}) dx$$

$$a^2 \pi \left(\frac{m_0}{2n+2} x^{2n+2} + \frac{c}{s+2n+2} x^{s+2n+2} \right)$$

(Perry, 76)

$$9. \int_x (a+x)(b-x) dx = \int_x [ab - (a-b)x - x^2]$$

$$= abx - \frac{1}{2}(a-b)x^2 - \frac{1}{3}x^3$$

$$10. \int (ax^m + b)a^1 x^n + b^1 dx = \int_x (aa^1 x^{m+n} + a^1 b x^n +$$

$$ab^1 x^m + b b^1)$$

$$= \frac{a a^1 x^{m+n+1}}{m+n+1} + \frac{a^1 b x^{n+1}}{n+1} + \frac{a b^1 x^{m+1}}{m+1} + b b^1 x.$$

$$11. \int_x (ax+b)(cx^2+px+q) = \frac{1}{12} x [3acx^3 + 4(ap+bc)x^2$$

$$+ 6(aq+bp)x + 12bq]$$

$$12. \int (\dot{x}^2 \dot{x} + a x \dot{x} + b b \dot{x}) = \frac{1}{3} x^3 + \frac{1}{2} a x^2 + b^2 x.$$

(Mac Laurin, II, 599)

En la notacion anterior, debida a Newton, $\dot{x} = d x$.

49. *Desarrollo de una potencia.*—Para integrar la potencia de un polinomio, tal como

$$(p_0 x^m + p_1 x^{m-1} + p_2 x^{m-2} + \dots + p_m)^n,$$

se desarrolla conforme a las reglas del Algebra i en seguida se integra cada uno de sus términos.

$$1. \int_x (a + b x)^2 = \int_x (a^2 + 2 a b x + b^2 x^2)$$

$$= \int_x a^2 + \int_x 2 a b x + \int_x b^2 x^2$$

$$= a^2 x + a b x^2 + \frac{1}{3} b^2 x^3.$$

$$2. \int (a x^2 - b)^2 = \frac{1}{6} a^2 x^6 - \frac{1}{2} a b x^4 + \frac{1}{2} b^2 x^2$$

(Timmermans, 247)

$$3. \int_x (1 - a x^2)^4 = x - \frac{4}{3} a x^3 + \frac{6}{5} a^2 x^5 - \frac{4}{7} a^3 x^7$$

$$+ \frac{1}{9} a^4 x^9.$$

4. $\int (1+x)^n dx$. Desarrollemos por el binomio de Newton:

$$(1+x)^n = 1 + \frac{n}{1!} x + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3$$

$$+ \frac{n(n-1)\dots(n-3)}{4!} x^4 + \dots + x^n.$$

Reemplazamos e integramos cada término:

$$\begin{aligned} \int (1+x)^n = x + \frac{n}{2!} x^2 + \frac{n(n-1)}{3!} x^3 + \frac{n(n-1)(n-2)}{4!} x^4 \\ + \frac{n(n-1)\dots(n-3)}{5!} x^5 + \dots + \frac{1}{n+1} x^{n+1}. \end{aligned}$$

5. $\int_x (a^n + x^n)^m$. Se puede proceder así:

$$(a^n + x^n)^m = a^{mn} \left[1 + \left(\frac{x}{a} \right)^n \right]^m.$$

$$= a^{mn} \left[1 + \frac{m}{1! a^n} x^n + \frac{m(m-1)}{2! a^{2n}} x^{2n} + \dots + \left(\frac{x}{a} \right)^{mn} \right]$$

$$\begin{aligned} \therefore \int_x (a^n + x^n)^m = a^{mn} \left[x + \frac{m}{1! a^n (n+1)} x^{n+1} \right. \\ + \frac{m(m-1)}{2! a^{2n} (2n+1)} x^{2n+1} + \dots \\ \left. + \frac{1}{a^{mn} (mn+1)} x^{mn+1} \right] \end{aligned}$$

$$\begin{aligned} 6. \int_x (a+bx+cx^2)^2 = a^2x + abx^2 + \frac{1}{3} (2ac+b^2) x^3 \\ + \frac{1}{2} bx^4 + \frac{1}{5} c^2 x^5. \end{aligned}$$

(Bossut, I, 253)

$$7. \int (a+bx+cx^2+\dots)^n dx$$

$$= a^n \int dx$$

$$+ \frac{n}{1!} a^{n-1} \int (bx + cx^2 + \dots)$$

$$+ \frac{n(n-1)}{2!} a^{n-2} \int (bx + cx^2 + \dots)^2$$

$$+ \frac{n(n-1)(n-2)}{3!} \int (bx + cx^2 + \dots)^3$$

$$+ \dots$$

$$8. \int x^2 (a-x)^3 dx$$

$$= \int (a^3 x^2 - 3a^2 x^3 + 3a x^4 - x^5) dx$$

$$= \frac{1}{2} a^3 x^3 - \frac{3}{4} a^2 x^4 + \frac{3}{5} a x^5 - \frac{1}{6} x^6$$

$$9. \int (a+bx^2)^3 (a-bx^2)^2 dx$$

$$= a^5 x - a^4 b x^3 - \frac{1}{2} a^4 b x^4 + \frac{3}{5} a^3 b^2 x^5 - a^3 b^2 x^6$$

$$+ \frac{1}{7} (a^2 b^3 + a^3 b^2) x^7 - \frac{3}{4} a^2 b^3 x^8 + \frac{1}{3} a^2 b^3 x^9$$

$$- \frac{1}{5} a b^4 x^{10} + \frac{3}{11} a b^4 x^{11} + b^5 x^{13}$$

$$10. \int \frac{a(x-b)^3}{n x^4} dx = \frac{a}{n} \int (x-b)^3 x^{-4} dx$$

$$= \frac{a}{n} \left(Lx + \frac{3b}{x} = \frac{3b^2}{2x^2} + \frac{b^3}{2x^3} \right)$$

(Pauly, 163)

$$11. \int_x (a x^n + b)^m x^r$$

$$= \frac{a^m x^{m n + r + 1}}{m n + r + 1}$$

$$+ \frac{m}{1!} \cdot \frac{a^{m-1} b}{n(m-1) + r + 1} x^{n(m-1) + r + 1}$$

$$+ \frac{m(m-1)}{2!} \cdot \frac{a^{m-2} b^2}{n(m-2) + r + 1} x^{n(m-2) + r + 1}$$

$$+ \dots$$

50. *Simplificaciones.*—Las fracciones, antes de integrarlas, deben ser reducidas a su más simple expresión, para lo cual se descomponen en factores sus dos términos.

$$1. \int_x \frac{x^2 - 1}{x - 1} = \int_x \frac{(x+1)(x-1)}{x-1} = \int (x+1) = \frac{1}{2} x^2 + x$$

$$2. \int_x \frac{1+x^3}{1+x} = \int_x (1 - x + x^3) = x - \frac{1}{2} x^2 + \frac{1}{3} x^3$$

$$3. \int_x \frac{x^2 - 3x + 2}{x - 1} = \int_x \frac{(x-1)(x+2)}{x-1} = \int_x (x+2)$$

$$= \frac{1}{2} x^2 + 2x$$

$$4. \int_x \frac{a^4 - a^4}{x^2 + a^2} = \int_x (x^2 + a^2) = \frac{1}{3} x^3 + a^2 x$$

$$5. \int_x \frac{x^5 - 1}{x - 1} = \int_x (x^4 + x^3 + x^2 + x + 1) \\ = \frac{1}{5} x^5 + \frac{1}{4} x^4 + \frac{1}{3} x^3 + \frac{1}{2} x^2 + x.$$

$$6. \int_x \frac{x^5 + a^5}{x^2 + a^2} = \int_x (x^3 - a^2 x + a^4) \\ = \frac{1}{5} x^5 + \frac{a^2}{3} x^2 + a^4 x$$

$$7. \int_x \frac{6x^2 - 8ax}{9ax - 12a^2} = \int_x \frac{2x(3x - 4a)}{4a(3x - 4a)} = \frac{2}{3a} \int_x = \frac{1}{3a} x^2$$

$$8. \int_x \frac{20(x^3 - a^3)}{(5x^2 + 2ax) + a(3x + 5a)} = \int_x 4(x - a) \\ = 4 \left(\frac{1}{2} x^2 - ax \right).$$

$$9. \int_x \frac{x^4 + x^3 + x^2 + x + 1}{x^5 - 1} = \int_x \frac{1}{x - 1} = L(x - 1)$$

51. *Descomponer por division.*—Para integrar una fracción cuyo numerador es de un grado superior o igual al del denominador, se efectúa la división.

$$1. \int_x \frac{x+1}{x-1} = \int_x \left(1 + \frac{2}{x-1} \right) = x + 2L(x-1)$$

$$2. \int_x \frac{x}{a+bx} = \int_x \left(\frac{1}{b} - \frac{a}{b(bx+a)} \right) = \frac{x}{b} - \frac{a}{b^2} L(ax+bx)$$

(Peacock, 269)

$$3. \int_x \frac{x^2}{x-1} = \int_x \left(x+1 + \frac{1}{x-1} \right) = \frac{1}{2}x^2 + x + L(x-1)$$

$$4. \int_x \frac{x^2+1}{x+1} = \int_x \left(x-1 + \frac{2}{x+1} \right) = \frac{1}{2}x^2 - x + 2L(x+1)$$

$$5. \int_x \frac{x^2-x+2}{x+1} = \int_x \left(x-2 + \frac{4}{x+1} \right)$$

$$= \frac{1}{2}x^2 - 2x + 4L(x+1)$$

$$6. \int_x \frac{x^2-2x+5}{x-1} = \int_x \left(x^2+x-1 + \frac{4}{x-1} \right)$$

$$= \frac{1}{3}x^3 + \frac{1}{2}x^2 - x + L(x-1)$$

(De Comberousse, IV, 684)

$$7. \int_x \frac{x^2+x-2}{x^2+2x-3} = \int_x \frac{x-2}{x-3} \int_x \left(1 - \frac{5}{x+3} \right)$$

$$= x + L(x+3)^5.$$

$$8. \int_x \frac{x^3}{a+bx} = \int_x \left[\frac{x^2}{b} - \frac{ax}{b^2} + \frac{a^2}{b^3} - \frac{a^3}{b^3(bx+a)} \right]$$

$$= \frac{x^3}{3b} - \frac{ax^2}{2b^2} + \frac{a^2x}{b^3} - \frac{a^3}{b^4} L(bx+a)$$

$$9. \int_x \frac{2-x^3}{1-x} = \frac{1}{3} x^3 + \frac{1}{2} x^2 + x - L(1-x)$$

(Bertrand, II, 131)

$$10. \int_x \frac{x^4}{a+bx^2} = \int_x \left[\frac{x^2}{b} - \frac{a}{b^2} + \frac{a^3}{b^2(bx^2+a)} \right] \\ = \frac{x^3}{3b} - \frac{ax}{b^2} + \frac{a^3}{b^2} \cdot \frac{1}{\sqrt{ab}} \operatorname{arc} \operatorname{tg} \sqrt{\frac{b}{a}} x.$$

$$11. \int_x \frac{x^5-1}{x+1} = \frac{1}{5} x^5 + \frac{1}{4} x^4 + \frac{1}{3} + \frac{1}{2} x^2 + x$$

$$12. \int_x \frac{\operatorname{sen}^2 x - \operatorname{sen} x - \cos x}{\operatorname{sen} x - 1} = -\cos x + L(\operatorname{sen} x - 1)$$

$$13. \int_x \frac{a^{x+1} - a^2 x - 1 - x a^x + x a^2 x - 1}{a-x} \int_x = (a^x - a^2 x - \frac{1}{a-x}) \\ = \frac{x^x}{La} - \frac{1}{2} a^2 x^2 + L(a-x)$$

$$14. \int_y \frac{y^n}{y-a} = \frac{y^n}{n} + \frac{a y_{n-1}}{n-1} + \frac{a y_{n-1}}{n-2} + \dots + a^{n-1} y \\ + \frac{a^n}{n} L(y-a). \quad (\text{Mac Laurin, II, 629})$$

$$15. \int_x \frac{\left(\frac{x}{a}\right)^3 + \left(\frac{a}{x}\right)^3 - 3\left[\left(\frac{x}{a}\right)^2 - \left(\frac{a}{x}\right)^2\right] + 4\left(\frac{x}{a} + \frac{a}{x}\right)}{\frac{x}{a} + \frac{a}{x}}$$

$$= \frac{x^3}{3a^2} - \frac{3x^2}{2a} - 3aLx + \frac{a^2}{x} + 3x$$

(R. Potts, V, 17)

52. *Descomposicion de las fracciones.*—La fraccion

$$\frac{F(x)}{f(x)} = \frac{ax^m + bx^{m-1} + \dots + p}{f(x)}$$

se descompone en las fracciones mas sencillas

$$\frac{F(x)}{f(x)} = a \frac{x^m}{f(x)} + b \frac{x^{m-1}}{f(x)} + \dots + \frac{p}{f(x)}$$

por simple division o segun los métodos que a continuacion se indican.

$$1. \int_x \frac{a+x^2}{x} = \int_x \left(\frac{a}{x} + x \right) = aLx + \frac{1}{2}x^2$$

$$2. \int_x \frac{x+1}{x^2+1} = \int_x \left(\frac{x}{x^2+1} + \frac{1}{x^2+1} \right) = \frac{1}{2}L(x^2+1)$$

+ arc tg x

(Axel Karnack, 190)

$$3. \int_x \frac{ax}{b+cx} = \int_x \frac{acx}{c(b+cx)} = \int_x \frac{acx+ab-ab}{c(b+cx)}$$

$$= \int_x \left(\frac{a}{c} - \frac{ab}{c(b+cx)} \right) = \frac{a}{c}x - \frac{ab}{c^2}L(b+cx)$$

(R. H. Smith, 61)

$$4. \quad \int_x \frac{(1+x^2)^2}{x} = \int_x \left(\frac{1}{x} + 12x^2 + x^5 \right) = Lx$$

$$+ \frac{2}{3} x^3 + \frac{1}{4} x^4$$

$$5. \quad \int_x \frac{(1-x^2)^2}{x} = Lx - x^2 + \frac{1}{4} x^4 \quad (\text{Williamson, 14})$$

$$6. \quad \int_y \frac{(y-a)^3}{y^m} = \int_y (y^{3-m} - 3ay^{2-m} + 3a^2 y^{1-m} - a^3 y^{-m})$$

$$= \frac{y^{4-m}}{4-m} - \frac{3ay^{3-m}}{3-m} + \frac{3a^2 y^{2-m}}{2-m} - \frac{a^3 y^{1-m}}{1-m}$$

(Perry, 281)

$$7. \quad \int_x \frac{a-bx}{a+x} = a L(c+x) - bx + c L(c+x)$$

(Véase ejercicio 3)

$$8. \quad \int_x \frac{1-x}{1+x^2} = \text{art tg } x - \frac{1}{2} L(1+x^2)$$

$$9. \quad \int_x \frac{1}{1-x^2} \quad \text{Esta integral se considera como funda-}$$

mental. La descomponemos como se indica:

$$\frac{1}{1-x^2} = \frac{1 + \frac{1}{2}x - \frac{1}{2}x}{1-x^2} = \frac{\left(\frac{1}{2} + \frac{1}{2}x\right) + \left(\frac{1}{2} - \frac{1}{2}x\right)}{(1+x)(1-x)}$$

$$= \frac{\frac{1}{2}(1+x) + \frac{1}{2}(1-x)}{(1+x)(1-x)} = \frac{1}{2} \left(\frac{1}{1+x} + \frac{1}{1-x} \right)$$

$$\int_x \frac{1}{1-x^2} = \frac{1}{2} \int_x \left(\frac{1}{1+x} + \frac{1}{1-x} \right)$$

$$= \frac{1}{2} L(1+x) - \frac{1}{2} L(1-x)$$

$$= L \sqrt{\frac{1+x}{1-x}}$$

$$10. \int_x \frac{1}{x^2-1} = L \sqrt{\frac{1-x}{1+x}}$$

$$11. \int_x \frac{1}{a^2-x^2} = \frac{1}{a} L \sqrt{\frac{a+x}{a-x}}$$

$$12. \int \frac{dx}{a^2-b^2x^2} = \frac{1}{2a} \int \frac{(a-bx) + (a+bx)}{(a+bx)(a-bx)}$$

$$= \frac{1}{ab} L \sqrt{\frac{a+bx}{a-bx}} \quad (\text{De Comberousse, 685})$$

$$13. \int_x \frac{ax+b}{x^2+c^2} = \int_x \frac{ax}{x^2+c^2} + \int_x \frac{b}{x^2+c^2}$$

$$= 4L \sqrt{a^2+c^2} + \frac{b}{c} \operatorname{arc} \operatorname{tg} \frac{x}{c} \quad (\text{Hotel, I, 213})$$

$$14. \int_x \frac{A \mp Bi}{x-a \mp bi} = \int_x \frac{(a \mp Bi)(x-a \pm bi)}{(x-a)^2+b^2}$$

$$= \frac{1}{2} (A \mp Bi) L[(x-a)^2 + b^2] + (B \pm ai) \operatorname{arc} \operatorname{tg} \frac{x-a}{b}$$

(Moigno, II, 17)

$$\begin{aligned} 15. \quad \int_x \frac{dx}{x^3-1} &= \int_x \frac{1}{3} \left(\frac{1}{x-1} - \frac{x+2}{x^2+x+1} \right) \\ &= \frac{1}{6} L \left[\frac{(x-1)^2}{x^2+x+1} \right] - \frac{1}{\sqrt{3}} \operatorname{arctg} \frac{2x+1}{\sqrt{3}} \end{aligned}$$

$$\begin{aligned} 16. \quad \int_x \frac{1+mx}{a+bx^2} &= 1 \int_x \frac{1}{a+bx^2} + m \int_x \frac{x}{a+bx^2} \\ &= \frac{1}{\sqrt{ab}} \operatorname{arc} \operatorname{tg} \sqrt{\frac{b}{a}} x + \frac{m}{2b} L(a+bx^2) \end{aligned}$$

(Roberts, 13)

$$17. \quad \int_x \frac{ax^2+bx+c}{px+q} = \int_x \left(\frac{ax^2}{px+q} + \frac{bx}{px+q} + \frac{c}{px+q} \right)$$

La integral se obtiene efectuando la division; las otras dos son conocidas:

$$\begin{aligned} y &= a \left[\frac{x^2}{2p} - \frac{q}{p^2} x + \frac{q^2}{p^3} L(px+q) \right] \\ &= b \left[\frac{1}{p} x - \frac{q}{p^2} L(px+q) \right] + \frac{c}{p} L(px+q). \end{aligned}$$

$$18. \quad \int_x \frac{Ax+B}{a+bx+cx^2} = \int_x \frac{ax + \frac{Ab}{2c} + B - \frac{Ab}{2c}}{a+bx+cx^2}$$

$$\begin{aligned}
&= \frac{A}{2c} \int_x \frac{2cx+b}{a+bx+cx^2} + \left(B - \frac{Ab}{2c} \right) \int_x \frac{1}{a+bx+cx^2} \\
&= \frac{A}{2c} L(a+bx+cx^2) + \left(B - \frac{Ab}{2c} \right) \frac{2}{\sqrt{4ac-b^2}} \operatorname{arctg} \frac{2cx+b}{\sqrt{4ac-b^2}} \\
&\quad (\text{Todhunter, 14})
\end{aligned}$$

$$\begin{aligned}
19. \quad \int_x \frac{ax+b}{x^2+px+q} &= \int_x \left(\frac{a}{2} \cdot \frac{2x+p-p}{x^2+px+q} + b \cdot \frac{1}{x^2+px+q} \right) \\
&= \left(b - \frac{ap}{2} \right) \int_x \frac{1}{x^2+px+q} + \frac{a}{2} \int_x \frac{2x+p}{x^2+px+q} \\
&= \left(b - \frac{ap}{2} \right) \frac{2}{\sqrt{4q-p^2}} \operatorname{art} \operatorname{tg} \frac{2x+p}{\sqrt{4q-p^2}} \\
&\quad + \frac{a}{2} L(x^2+px+q). \quad (\text{Gregory, 251})
\end{aligned}$$

$$\begin{aligned}
20. \quad \int_x \frac{3x^4-6x^2+1}{x^2-2x+2} &= \int_x \left(3x^2+6x + \frac{1-12x}{x^2-2x+2} \right) \\
&= x^3+3x^2-6L(x^2-2x+2)+11 \operatorname{arc} \operatorname{tg} (x-1). \\
&\quad (\text{Timmermans, 259})
\end{aligned}$$

$$21. \quad \int \frac{Ax+B}{(x-a)^2+\beta^2} dx. \text{ Agreguemos } a-a=0 \text{ al numerador:}$$

$$\frac{Ax+B}{(x-a)^2+\beta^2} = A \frac{x+a-a}{(x-a)^2+\beta^2} + \frac{B}{(x-a)^2+\beta^2}$$

$$= \frac{A}{2} \cdot \frac{2(x-a)}{(x-a)^2 + \beta^2} + \frac{B + A a}{(x-a)^2 + \beta^2}$$

$$\therefore y = \frac{1}{2} A L[(x-a)^2 + \beta^2] + \frac{B + A a}{\beta} \operatorname{arc} \operatorname{tg} \frac{x-a}{\beta}$$

(Briot, II, 503)

$$\begin{aligned} 22. \int \frac{x+1}{x \sqrt{1-x^2}} &= \int \frac{x}{x \sqrt{1-x^2}} + \int \frac{1}{x \sqrt{1-x^2}} \\ &= -\sqrt{1-x^2} + \operatorname{arc} \operatorname{sen} x. \end{aligned}$$

(Moigno, II, 15)

$$\begin{aligned} 23. \int \frac{x dx}{(x^2 + p x + q)^{\frac{3}{2}}} &= \frac{1}{2} \int \frac{2x + p}{(x^2 + p x + q)^{\frac{3}{2}}} dx \\ &\quad - \frac{p}{2} \int \frac{dx}{(x^2 + p x + q)^{\frac{3}{2}}} \\ &= (x^2 + p x + q)^{-\frac{1}{2}} - \frac{2}{4q - p^2} \cdot \frac{p x + 2q}{\sqrt{x^2 + p x + q}} \end{aligned}$$

(Brahya, 24)

24. $\int e^x \cdot \frac{x^2 + 1}{(x+1)^2} dx$. La fracción se descompone así:

$$\begin{aligned} \frac{x^2 + 1}{(x+1)^2} &= \frac{x^2 + 1 - 1 + 1}{(x+1)^2} = \frac{x^2 + 1}{(x+1)^2} + \frac{2}{(x+1)^2} \\ &= \frac{x-1}{x+1} + \frac{2}{(x+1)^2} \end{aligned}$$

Pero d $\frac{x-1}{x+1} = \frac{2}{(x-1)^2}$

$$y = \int e^x \left(\frac{x-1}{x+1} dx + d \frac{x-1}{x+1} \right)$$

$$= \int \left(\frac{x-1}{x+1} d e^x + e^x d \frac{x-1}{x+1} \right)$$

$$= \int (u dv + du) = \int d(uv)$$

$$= e^x \cdot \frac{x-1}{x+1} \quad (\text{Grégory, 253})$$

$$25. \int_x \frac{x \cdot e^x}{(1+x)^2} = \int_x e^x \left[\frac{x+1-1}{(1+x)^2} \right]$$

$$= \int_x e^x \left(\frac{dx}{1+x} + d \frac{1}{1+x} \right)$$

$$= \int_x \left(\frac{d e^x}{1+x} + e^x d \frac{1}{1+x} \right)$$

$$= e^x \frac{1}{1+x}$$

$$26. \int_x e^x \frac{2-x^2}{\sqrt{1+x} \sqrt{(1-x)^3}} = \int_x e^x \left[\frac{1+(1+x)(1-x)}{\sqrt{(1+x)(1-x)}(1-x)^2} \right]$$

$$= \int_x \left(d e^x \sqrt{\frac{1+x}{1-x}} + e^x \sqrt{\frac{1+x}{1-x}} \right) = e^x \sqrt{\frac{1+x}{1-x}}$$

$$27. \int \frac{dx}{x L x} = \int_x \frac{1 + L x - L x}{x L x} \int_x \frac{1 + L x}{x L x} \int \frac{dx}{x}$$

$$= L(x L x) - L x = L(L x)$$

(Perry, 284)

$$28. \int \frac{1 - x \sin a}{1 - 2x \sin a + x^2} dx. \text{ Hacemos } 1 = \sin^2 a + \cos^2 a:$$

$$y = \int \frac{\cos^2 a - \sin a (x - \sin a)}{1 - 2x \sin a + x^2} dx$$

$$= \cos a \cdot \operatorname{arc} \operatorname{tg} \frac{x - \sin a}{\cos a} - \sin a L \sqrt{1 - 2x \sin a + x^2}$$

(Brahya, 12)

53. *Fracciones parciales.*—Son las que resultan de la descomposición de una fracción común en otras fracciones, cuya suma es igual a la primera.

$$1. \int \frac{1}{1-x^2} dx \text{ Integral importante i conocida.}$$

Para descomponerla en otras dos, descomponemos el denominador en sus factores:

$$1-x^2 = (1+x)(1-x);$$

en seguida nos valemos de las *indeterminadas* A, B, de modo que verifiquen la igualdad:

$$\frac{1}{(1+x)(1-x)} = \frac{A}{1+x} + \frac{B}{1-x};$$

hacemos desaparecer los denominadores, i se obtiene:

$$1 = A(1-x) + B(1+x). \quad (1)$$

Como x es una variable, podemos suponer $x=1$ i resulta

$$B = \frac{1}{2}; \text{ i si suponemos } x = -1, \text{ resulta } A = \frac{1}{2} \text{ luego,}$$

$$\frac{1}{(1+x)(1-x)} = \frac{\frac{1}{2}}{1+x} + \frac{\frac{1}{2}}{1-x}.$$

$$\int_x \frac{1}{1-x^2} = \frac{1}{2} \int \frac{1}{1+x} + \frac{1}{2} \int \frac{1}{1-x}$$

$$= \frac{1}{2} L(1+x) - \frac{1}{2} L(1-x)$$

$$= L \sqrt{\frac{1+x}{1-x}}.$$

Método de los coeficientes indeterminados.—La ecuacion (1) se escribe así:

$$1 = (A+B) + (B-A)x;$$

o bien,

$$x^0 + 0 \cdot x^1 = (A+B)x^0 + (B-A)x^1$$

En Algebra superior se demuestra que esta relacion es cierta cuando son iguales los coeficientes de las mismas potencias de x :

$$\left. \begin{array}{l} A+B=1 \\ B-A=0 \end{array} \right\} \therefore B = \frac{1}{2}, A = \frac{1}{2}$$

$$2. \int \frac{1}{x^2-1} dx = \int \left(\frac{A}{x+1} + \frac{B}{x-1} \right) dx = L \sqrt{\frac{x-1}{x+1}}$$

$$3. \int \frac{1}{a^2-x^2} dx = \int \left(\frac{A}{a+x} + \frac{B}{a-x} \right) dx = \frac{1}{2a} L \frac{a+1}{a-x}$$

o bien se reduce a la forma elemental:

$$\int \frac{dx}{a^2-x^2} = \frac{1}{a} \int \frac{\frac{x}{a}}{1-\left(\frac{x}{a}\right)^2} = \frac{1}{a} L \sqrt{\frac{a+x}{a-x}}$$

$$4. \int \frac{1}{x^2-a^2} dx = \frac{1}{a} \int \frac{\frac{x}{a}}{\left(\frac{x}{a}\right)^2-1} = \frac{1}{a} L \sqrt{\frac{x-a}{x+a}}$$

$$5. \int \frac{a}{a^2x^2-b^2} dx = \int \frac{1}{2b} L \frac{ax-b}{ax+b}$$

$$6. \int \frac{1}{(x-a)(x-b)} dx = \int \left(\frac{A}{x-a} + \frac{B}{x-b} \right) dx; A = \frac{1}{a-b}$$

$$B = \frac{1}{b-a} \therefore y = \frac{1}{a-b} L \frac{x-a}{x-b}$$

(Williamson, II, 8)

$$7. \int \frac{dx}{a+2bx+cx^2} = \int \frac{dz}{z^2+a^2-b^2} \quad (z=cx+b)$$

Si $a c - b^2$ es positivo,

$$y = \frac{1}{\sqrt{a c - b^2}} \operatorname{arc} \operatorname{tg} \frac{c x + b}{\sqrt{a c - b^2}};$$

i si es negativo:

$$y = \frac{1}{2 \sqrt{b^2 - a c}} L \frac{c x + b - \sqrt{b^2 - a c}}{c x + b + \sqrt{b^2 - a c}}$$

8. $\int \frac{d x}{x^2 + 9 x + 20}$. Descompongamos el trinomio en sus factores:

$$x^2 + 9 x + 20 = (x + 4)(x + 5);$$

luego,

$$\frac{1}{x^2 + 9 x + 20} = \frac{A}{x + 4} + \frac{A}{x + 5}$$

$$1 = A(x + 5) + B(x + 4)$$

Sea

$$x = -4 \dots A = 1; x = -5 \dots B = -1$$

$$\therefore \int \frac{d x}{x^2 + 9 x + 20} = L \frac{x + 4}{x + 5}$$

$$9. \frac{a x - b}{x^2 - c^2} = \frac{A}{x + c} + \frac{B}{x - c}$$

$$\therefore ax - b = A(x - c) + b(x + c)$$

Sea

$$x = c \quad \therefore ac - b = 2Bc \quad \therefore B = \frac{ac - b}{2c};$$

Sea

$$x = -c \quad \therefore -ac - b = -2Ac \quad \therefore A = \frac{ac + b}{2c};$$

$$\therefore \int \frac{ax - b}{x^2 - c^2} dx = \frac{ac + b}{2c} L(x + c) + \frac{ac - b}{2c} L(x - c)$$

(Timmermans, 261)

$$10. \quad \frac{4x - 1}{3x^2 - 15x + 8} = \frac{1}{3} \left(\frac{A}{x - 2} + \frac{B}{x - 3} \right)$$

$$\therefore 4x - 1 = A(x - 3) + B(x - 2).$$

Sea

$$x = 2, \quad 8 + 1 = A \quad \therefore A = 9;$$

Sea

$$x = 3, \quad 12 + 1 = B \quad \therefore B = 13:$$

$$\frac{4x - 1}{3x^2 - 15x + 8} dx = L \frac{(x - 3) \frac{13}{3}}{(x - 2)^3}$$

(Sonnet, 207)

$$11. \int \frac{(2-4x) dx}{x^2-x-2} = \int \frac{2 dx}{2-x} - \int \frac{2 dx}{x+1}$$

$$= L \frac{1}{(x^2-x-2)^2}$$

(Francoeur, 360)

$$12. \frac{4x+1}{3(x-2)^2} = \frac{1}{3} \left[\frac{A}{x-2} + \frac{B}{(x-2)^2} \right]$$

$$\therefore 4x+1 = A(x-2) + B = Ax - 2A + B$$

$$\therefore A=4, B-2A=1 \quad \therefore B=9$$

$$\therefore y = \frac{4}{3} L(x-2) - \frac{1}{x-2}$$

13. $\frac{mx+n}{ax^2+bx+c}$. Sean α i β las raíces reales i diferentes del trinomio:

$$\frac{mx+n}{(x-\alpha)(x-\beta)} = \frac{A}{x-\alpha} + \frac{B}{x-\beta}$$

$$\therefore mx+n = A(x-\beta) + B(x-\alpha)$$

Sea

$$x=\alpha, \text{ queda: } m\alpha+n = A(\alpha-\beta)$$

$$\therefore A = \frac{m\alpha+n}{\alpha-\beta}$$

Sea

$$x = \beta, \text{ queda: } m\beta + n = B(\beta - a)$$

$$B = \frac{m\beta + n}{a - \beta}$$

$$y = \frac{m\alpha + n}{a - \beta} L(x - a) + \frac{m\beta + n}{a - \beta} L(x - \beta).$$

(Pauly, 165)

14. $\frac{2x-3}{x^2+2ax+3a^2}$. Agreguemos $2a-2a=0$; la fracción se descompone así:

$$\frac{2x+2a}{x^2+2ax+3a^2} - \frac{2a+3}{x^2+2ax+3a^2}.$$

El trinomio se descompone en

$$2a^2 + (x+a)^2.$$

La integral es,

$$y = L(x^2 + 2ax + 3a^2) - \frac{2a+3}{a\sqrt{2}} \arctg \frac{x+a}{a\sqrt{2}}$$

(Brahya, 9)

15. $\frac{5x^3+1}{x^2-3x+2}$. Efectuamos la división y se obtiene el cociente:

$$5x+15 + \frac{35x-29}{x^2-3x+2}$$

Descompongamos la nueva fraccion:

$$35x-29 = A(x-2) + B(x-1)$$

$$A = -6, B = 41$$

$$\therefore y = \int \left(5x+15 - \frac{6}{x-1} + \frac{41}{x-2} \right) dx$$

$$= \frac{5}{2} x^2 + 15x - 6L(x-1) + 41L(x-2)$$

(Todhunter, II, 31)

$$16. \frac{1}{x^3-x^2-x+1} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

$$\therefore 1 = A(x^2-1) + B(x+1) + C(x-1)^2$$

$$\text{sea } x=1, B(1+1)=1 \therefore B = \frac{1}{2}$$

$$\text{sea } x=-1, C(-1-1)^2 = 1 \therefore C = \frac{1}{4}$$

Diferenciemos ahora:

$$0 = 2Ax + B + 2C(x-1)$$

$$\text{sea } x=1 \therefore 0 = 2A + B = 2A + \frac{1}{2} \therefore A = -\frac{1}{4}$$

$$\therefore y = L \left(\frac{x+1}{x-1} \right)^{\frac{1}{4}} - \frac{1}{2} \frac{1}{x-1}$$

(Grégory, 262)

$$17. \quad \frac{x}{x^3-1} = \frac{Ax+B}{x^2+x+1} + \frac{C}{x-1}$$

$$\therefore x = (A+C)x^2 + (B+C-A)x + C-B:$$

$$A+C=0, B+C-A=1, C-B=0$$

$$\therefore \int \frac{x}{x^3-1} dx = \frac{1}{3} \left[L(x-1) - L \sqrt{x^2+x+1} + \sqrt{3} \arctan \left(\frac{2x+1}{\sqrt{3}} \right) \right]$$

(Francoeur, 361)

$$18. \quad \frac{du}{dx} = \frac{x-3}{x^3+1} = \frac{x-3}{(x+1)(x^2-x+1)}$$

Sea

$$\frac{x-3}{x^3+1} = \frac{A}{x+1} + \frac{Mx+N}{x^2-x+1}$$

$$\therefore x-3 = A(x^2-x+1) + (x+1)(Mx+N)$$

Sea

$x = -1$, se obtiene $A = -\frac{4}{3}$, valor que sustituimos arriba:

$$x-3 = -\frac{4}{3}(x^2-x+1) + (x+1)(Mx+N)$$

$$\therefore Mx+N = \frac{4x-5}{2}$$

$$\int_x \frac{x-3}{x^3+1} = -\frac{4}{3} \int_x \frac{1}{x+1} + \frac{1}{3} \int_x \frac{4x-5}{x^2-x+1}$$

La primera integral es: $-\frac{4}{3} L(x+1)$; i la segunda se transforma así:

$$\begin{aligned} \int \frac{4x-5}{\left(x-\frac{1}{2}\right)^2 + \frac{3}{4}} &= \int \frac{4x}{\left(x-\frac{1}{2}\right)^2 + \frac{3}{4}} - \int \frac{5}{\left(x-\frac{1}{2}\right)^2 + \frac{3}{4}} \\ &= 2L\left[\left(x-\frac{1}{2}\right)^2 + \frac{3}{4}\right] - 2\sqrt{3} \operatorname{arc} \operatorname{tg} \frac{2x-1}{\sqrt{3}}. \end{aligned}$$

(Hall, 228)

$$19. \frac{2x+3}{x^3+x^2-2x} = \frac{5}{3} \cdot \frac{1}{x-1} - \frac{1}{6} \cdot \frac{1}{x+2} - \frac{3}{2} \cdot \frac{1}{x}$$

$$\therefore y = L \frac{(x-1)^{\frac{5}{3}}}{x^{\frac{3}{2}} (x+2)^{\frac{1}{6}}}$$

$$20. \frac{du}{dx} = \frac{x^2-7x+1}{x^3-6x^2+11x+6}. \text{ Hacemos sucesivamen-}$$

te $x=1, 2, 3$ i el denominador se reduce cada vez a cero. luego, la fraccion se descompone así:

$$\frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3};$$

$$\begin{aligned} \therefore x^2 - 7x + 1 &= A(x-2)(x-3) + B(x-1)(x-3) \\ &\quad + C(x-1)(x-2) \end{aligned}$$

Sea $x=1, 2, 3$; se obtiene sucesivamente:

$$A = -\frac{5}{2}, \quad B = 9, \quad C = -\frac{11}{2}$$

$$\therefore u = L \frac{(x-2)^9}{\sqrt{(x-1)^6(x-3)^{11}}}$$

21. $\int \frac{x^2 - 3x + 5}{x^3 + x^2 - 4x - 4} dx$. Los factores del denominador

son $x+1, x-2, x+2$; i las fracciones parciales:

$$\frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{x+2}$$

$$x^2 - 3x + 5 = A \begin{vmatrix} x^2 & + & B \begin{vmatrix} x^2 & + & C \begin{vmatrix} x^2 \\ + 4 \end{vmatrix} \begin{vmatrix} x^2 \\ + 3x \\ + 2 \end{vmatrix} \begin{vmatrix} x^2 \\ -x \\ -2 \end{vmatrix} \end{vmatrix}$$

Igualemos los coeficientes de las potencias iguales de x :

$$\begin{aligned} A+B+C &= 1 \\ 3B-C &= -3 \\ -4A+2B-2C &= 5 \end{aligned} \quad A = \frac{\begin{vmatrix} 1 & 1 & 1 \\ -3 & 3 & -1 \\ 5 & 2 & -2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 0 & 3 & -1 \\ -4 & 2 & -2 \end{vmatrix}} = 3,$$

$$B = \frac{1}{4}, C = \frac{15}{4}$$

$$\therefore = -3 L(x+1) + \frac{1}{4} L(x-2) + \frac{15}{4} L(x+2)$$

(Pauly, 168)

$$22. \frac{3x^2 - 2x + 1}{x^4 + 2x^3 - 5x^2 - 6x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-2} + \frac{D}{x+3}$$

$$\therefore 3x^2 + 2x + 1 = A \begin{vmatrix} x^3 \\ + 2x^2 \\ - 5x \\ - 6 \end{vmatrix} + B \begin{vmatrix} x^3 \\ + x^2 \\ - 6x \end{vmatrix} + C \begin{vmatrix} x^3 \\ + 4x^2 \\ + 3x \end{vmatrix} + D \begin{vmatrix} x^3 \\ - x^2 \\ - 2x \end{vmatrix}$$

Igualamos los coeficientes de las mismas potencias de x:

$$\begin{array}{lcl} A + B + C + D & = 0 & \therefore A = -\frac{1}{6} \\ 2A + B + 4C - D & = 3 & B = 1 \\ -5A - 6B + 3C - 2D & = -2 & C = \frac{3}{10} \\ -6A & = 1 & D = \frac{17}{15} \end{array}$$

$$\therefore y = \frac{1}{6} Lx + L(x+1) + \frac{3}{10} L(x-2) - \frac{17}{15} L(x+3)$$

$$= L \frac{(x+1)(x-2)^{\frac{3}{10}}}{\sqrt[6]{x} + \sqrt[15]{(x+3)^{17}}}$$

(Timmermans, 262)

$$23. \quad \frac{1}{x^5 + x^4 + 2x^3 + 2x^2 + x + 1} = \frac{A(x-1)}{(x^2+1)^2} + \frac{B(x-1)}{x^2+1}.$$

$$\therefore y = \frac{1}{4} \cdot \frac{x+1}{x^2+1} + \frac{1}{2} \operatorname{arc} \operatorname{tg} x + \frac{1}{4} L \frac{x+1}{\sqrt{x^2+1}}$$

(Grégory, 263)

$$24. \quad \frac{x^3 + x^2 + 2}{x^5 - 2x^3 + x} dx$$

$$= \frac{2dx}{x} + \frac{dx}{(x-1)^2} - \frac{\frac{3}{4} dx}{x-1} - \frac{\frac{1}{2} dx}{(x+1)^2} - \frac{\frac{5}{4} dx}{x+1}$$

$$\therefore y = 2Lx - \frac{1}{x-1} - \frac{3}{4}L(x-1) + \frac{1}{2(x+1)} - \frac{5}{4}L(x+1).$$

$$25. \quad \frac{2x^2+1}{(x-2)^3(x+3)^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{(x+2)^3}$$

$$+ \frac{D}{x+3} + \frac{E}{(x+3)^2}$$

$$\therefore 2x^2+1 = A(x-2)^2(x+3)^2$$

$$+ B(x-2)(x+3)^2$$

$$+ C(x+3)^2$$

$$+ D(x-2)^3(x+3)$$

$$+ E(x-2)^3.$$

Sea

$$x=2, \text{ se obtiene: } 8+1=C(2+3)^2 \therefore C = \frac{9}{25}$$

Sea

$$x=-3, \text{ se obtiene: } +18+1=E \cdot -5^3 \therefore E = -\frac{19}{125}$$

Para encontrar los demas coeficientes, diferenciamos los dos miembros:

$$4x = A [2(x-2)(x+3)^2 + 2(x-2)^2(x+3)]$$

$$+ B [(x+3)^2 + 2(x-2)(x+3)]$$

$$+ C \cdot 2(x+3)$$

$$+ D [3(x-2)^2(x+3) + (x-2)^3]$$

$$+ E \cdot 3(x-2)^2$$

Sea, ahora,

$$x=2: 8=25B+10E=25B+\frac{90}{25}$$

$$\therefore B = \frac{22}{125}$$

Sea

$$x=-3: -12=D(-5)^3+E \cdot 3 \cdot 25=-125E-\frac{57}{5}$$

$$\therefore D = \frac{3}{6250}$$

Diferenciemos de nuevo, conservando sólo los términos que no se anulan para $x=2$:

$$4 = A \cdot 2(x+3)^2 + B[2(x+3) + 2(x+3)] + C \cdot 2.$$

Sea

$$x=2: 4=50A+20B+2C \quad \therefore A = -\frac{3}{625}$$

$$\therefore y = -\frac{3}{5^4(x-2)} + \frac{22}{5^3(x-2)^2} + \frac{9}{25(x-2)^3} \\ + \frac{3}{5^4(x+3)} - \frac{19}{5^3(x+3)^2}.$$

(Homersham Cox, 55)

$$26. \quad \frac{x^3 - x^2 + x - 1}{(x-1)^3(x+2)^2(x+1)} = \frac{A}{(x-1)^3} + \frac{B}{(x-1)^2} + \frac{C}{x-1} \\ + \frac{D}{(x+2)^2} + \frac{E}{x+2} + \frac{F}{x+1}$$

$$\therefore x^3 - x^2 + x - 1 = A(x+2)^2(x+1)$$

$$+ B(x-1)(x+2)^2(x+1)$$

$$+ C(x-1)^2(x+2)^2(x+1)$$

$$+ D(x-1)^3(x+1)$$

$$+ E (x-1)^3(x+2) (x+1)$$

$$+ F (x-1)^3(x+2)^2$$

Sea

$$x = -1, \text{ se obtiene } F = \frac{1}{2}; \text{ para } x = -2, \text{ resulta } D = \frac{5}{9}$$

Diferenciando, se determinan los demas coeficientes

$$27. \quad y = \int \frac{dt}{(1-t^2)^2} \quad \text{Aquí } (1-t^2)^2 = [(1+t)(1-t)]^2$$

$$\frac{1}{(1-t^2)^2} + \frac{A}{1+t} + \frac{B}{1-t} + \frac{C}{(1+t)^2} + \frac{D}{(1-t)^2}$$

$$\therefore 1 = A (1-t)^2(1+t) + B (1+t)^2(1-t)$$

$$+ C (1-t)^2 + D (1+t)^2.$$

Sea

$$x=1, \quad D (1+1)^2 = 1 \quad \therefore D = \frac{1}{4}$$

Sea

$$x=-1, \quad C (1+1)^2 = 1 \quad \therefore C = \frac{1}{4}$$

Para encontrar los demas coeficientes, diferenciamos:

$$0 = A (1-t)^2 - 2 A (1+t) (1-t) - B (1+t)^2$$

$$+ 2B(1-t)(1+t) - 2C(1-t) + 2D(1+t).$$

Sea

$$x=1; \quad -B(1+1)^2 + 2D(1+1) = 0 \quad \therefore B = \frac{1}{4}$$

Sea

$$x=-1, \quad A(1+1)^2 - 2C(1+1) = 0 \quad \therefore A = \frac{1}{4}$$

$$\begin{aligned} \therefore y &= \frac{1}{4} \int \left[\frac{1}{1+t} + \frac{1}{1-t} + \frac{1}{(1+t)^2} + \frac{1}{(1-t)^2} \right] \\ &= \frac{1}{4} \left[L(1+t) - L(1-t) + \frac{1}{1-t} - \frac{1}{1+t} \right] \\ &= \frac{1}{4} \left[L \frac{1+t}{1-t} + \frac{2t}{1-t^2} \right] \end{aligned}$$

$$28. \quad y = \int \frac{dx}{1-x^6} \quad (\text{Euler, I, 40})$$

$$\frac{dx}{1-x^6} = \frac{A}{1+x} + \frac{B}{1-x} + \frac{C+Dx}{1+x+x^2} + \frac{E+Fx}{1-x+x^2}$$

Hagamos desaparecer los denominadores.

$$1 = A(1-x)(1+x+x^2)(1-x+x^2)$$

$$+ B(1+x)(1+x+x^2)(1-x+x^2)$$

$$+ (C+Dx)(1-x^2)(1-x+x^2)$$

$$+(E+Fx)(1-x^2)(1+x+x^2)$$

Sea

$$x=1, \quad 1=B(1+1)(1+1+1)(1-1+1) \dots B=\frac{1}{6}$$

$$x=-1, \quad 1=A(1+1)(1-1+1)(1+1+1) \dots A=\frac{1}{6}$$

Procediendo como en los ejercicios anteriores encontramos que

$$\frac{1}{1-x^3} = \frac{\frac{1}{2}}{1+x} + \frac{\frac{1}{6}}{1-x} + \frac{\frac{1}{3} + \frac{1}{6}x}{1+x+x^2} + \frac{\frac{1}{3} + \frac{1}{6}x}{1-x+x^2}$$

$$\therefore y = \frac{1}{6} L \frac{1+x}{1-x} \sqrt{\frac{1+x+x^2}{1-x+x^2}} + \frac{1}{2\sqrt{3}} \operatorname{arc} \operatorname{tg} \frac{\sqrt{3}x}{1-x^2}$$

(Grégory, 264)

$$= -\frac{1}{6} L(1-x) - \frac{1}{12} L(1-x+x^2) + \frac{1}{2\sqrt{3}} \operatorname{arc} \operatorname{tg} \frac{x\sqrt{3}}{2-x}$$

$$+ \frac{1}{6} L(1+x) + \frac{1}{12} L(1+x+x^2) + \frac{1}{2\sqrt{3}} \operatorname{arc} \operatorname{tg} \frac{x\sqrt{3}}{2+x}$$

(Peacock, 279)

$$29. \quad \frac{1}{x^n(x-1)^n} = \frac{1}{x^n} + \frac{1}{(1-x)^n} + n \left[\frac{1}{x^{n-1}} + \frac{1}{(1-x)^{n-1}} \right]$$

$$+ \frac{n(n+1)}{2!} \left[\frac{1}{x^{n-2}} + \frac{1}{(1-x)^{n-2}} \right] + \dots$$

$$\frac{n(n+1)(n+2) \dots 2(n-1)}{(n-1)!} \left[\frac{1}{x} + \frac{1}{1-x} \right]$$

$$y = \frac{1}{n-1} \left[\frac{1}{(1-x)^{n-1}} - \frac{n}{x^{n-1}} \right] + \frac{n}{n-2} \left[\frac{n}{(1-x)^{n-2}} - \frac{1}{x^{n-2}} \right]$$

$$+ \dots + \frac{n(n+1) \dots 2(n-1)}{(n-1)!} L \frac{1}{1-x}$$

(Murphy, VI)

54. *Descomposicion de radicales.*—Para descomponer en un binomio el radical $\sqrt{1+x^2}$, se multiplica por $1 = \sqrt{1+x^2} : \sqrt{1+x^2}$:

$$\sqrt{1+x^2} = \frac{1+x^2}{\sqrt{1+x^2}} = \frac{1}{\sqrt{1+x^2}} + \frac{x^2}{\sqrt{1+x^2}}.$$

De igual manera se obtiene:

$$\sqrt{a^2 - x^2} = \frac{a^2}{\sqrt{a^2 - x^2}} - \frac{x^2}{\sqrt{a^2 - x^2}},$$

$$\sqrt{a+bx+cx^2} = \frac{a}{\sqrt{a+bx+cx^2}} + \frac{b}{\sqrt{a+bx+cx^2}}$$

$$+ \frac{cx^2}{\sqrt{a+bx+cx^2}}.$$

En la integracion por partes se hará principalmente esta descomposicion.

$$1. \int_x \sqrt{\frac{1+x}{1-x}} = \int_x \frac{1+x}{\sqrt{1-x^2}} = \int_x \left(\frac{1}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}} \right)$$

La primera integral es conocida, i es igual a

$$\text{arc sen } x.$$

La segunda se obtiene asi:

$$\int_x \frac{x}{\sqrt{1-x^2}} = -\frac{1}{2} \int (1-x^2)^{-\frac{1}{2}} \cdot -2 dx = -\sqrt{1-x^2}$$

$$\therefore \int \sqrt{\frac{1+x}{1-x}} \text{ arc sen } x - \sqrt{1-x^2}$$

(Euler, IV, 10)

$$2. \int_x \frac{\sqrt{a-x}}{\sqrt{a+x}} = \int_x \frac{a}{\sqrt{a^2-x^2}} - \int_x \frac{x}{\sqrt{a^2-x^2}} \\ = a \text{ arc sen } \frac{x}{a} + \sqrt{a^2-x^2}$$

$$3. \int \frac{dx}{\sqrt{x+a} + \sqrt{x+b}} = \int_x \frac{\sqrt{x+a} - \sqrt{x+b}}{x+a - (x+b)} \\ = \frac{1}{a-b} \int_x (\sqrt{x+a} - \sqrt{x+b})$$

$$= \frac{2}{3(a-b)} \left[(x+b)^{\frac{2}{3}} - (x+b)^{\frac{3}{2}} \right]$$

(Grégory, 252)

$$\begin{aligned} 4. \quad \int \frac{\sqrt{x+a}}{x \sqrt{x-a}} dx &= \int \frac{dx}{\sqrt{x^2-a^2}} + \int \frac{a dx}{x \sqrt{x^2-a^2}} \\ &= \operatorname{arc} \sec \frac{x}{a} + L \frac{x + \sqrt{x^2-a^2}}{a} \end{aligned}$$

$$\begin{aligned} 5. \quad \int \frac{\sqrt{x^2-a^2}}{x} dx &= \int \frac{x}{\sqrt{x^2-a^2}} - \int \frac{a^2}{x \sqrt{x^2-a^2}} \\ &= \sqrt{x^2-a^2} - a \cdot \operatorname{arc} \sec \frac{x}{a} \end{aligned}$$

$$6. \quad \int \frac{\sqrt{x^2+a^2}}{x} dx = \sqrt{x^2+a^2} + a L \frac{x}{\sqrt{x^2+a^2}+a}$$

$$\begin{aligned} 7. \quad \int \frac{\sqrt{x+1}}{x \sqrt{x-1}} dx &= \int \frac{dx}{\sqrt{x^2-1}} + \frac{dx}{\sqrt{x^2-1}} \\ &= L(x + \sqrt{x^2-1}) + \operatorname{arc} \sec x. \end{aligned}$$

(Brahý, 20)

55. *Descomposicion trigonométrica*, o sea descomponer la derivada trigonométrica en dos o mas términos.

$$1. \quad \int \operatorname{versen} x dx = \int (1 - \cos x) dx$$

$$= \int dx - \int \cos x dx = x - \operatorname{sen} x$$

$$2. \int \cos x \operatorname{sen} x dx = \int (1 - \operatorname{sen}^2 x) dx = x - \cos x.$$

$$3. \int \operatorname{sen}^2 x dx = \int \frac{1 - \cos 2x}{2} dx$$

$$= \frac{1}{2} \int dx = \frac{1}{4} \int \cos 2x dx$$

$$= \frac{1}{2} x - \frac{1}{4} \operatorname{sen} 2x$$

$$4. \int \cos^2 x dx = \int \frac{1 + \cos 2x}{2} dx = \frac{1}{2} x + \frac{1}{4} \operatorname{sen} 2x.$$

5. Otro método. (Humbert, I, 199).

$$\int_x \cos^2 x + \int_x \operatorname{sen}^2 x = \int_x (\cos^2 x + \operatorname{sen}^2 x) = \int dx = x,$$

$$\int_x \cos^2 x - \int_x \operatorname{sen}^2 x = \int_x (\cos^2 x - \operatorname{sen}^2 x) = \int_x \cos 2x = \frac{1}{2} \operatorname{sen} 2x,$$

sumando i restando, se obtiene:

$$\int \cos^2 x dx = \frac{1}{2} x + \frac{1}{4} \operatorname{sen} 2x \quad (\text{Daries, 116})$$

$$\int \operatorname{sen}^2 x dx = \frac{1}{2} x - \frac{1}{4} \operatorname{sen} 2x \quad (\text{Bowser, 246})$$

$$\begin{aligned}
 6. \quad \int \operatorname{tg}^2 x \, dx &= \int \frac{\operatorname{sen}^2 x}{\cos^2 x} \, dx = \int \frac{1 - \cos^2 x}{\cos^2 x} \, dx \\
 &= \int \frac{dx}{\cos^2 x} - \int dx = \operatorname{tg} x - x.
 \end{aligned}$$

(Sonnet, 232)

7. Otro método. (Frenet, 252).

$$\int \operatorname{tg}^2 x \, dx = \int (\sec^2 x - 1) \, dx = \operatorname{tg} x - x.$$

$$\begin{aligned}
 8. \quad \int \cot^2 x \, dx &= \int \frac{\cos^2 x}{\operatorname{sen}^2 x} = \int \frac{1 - \operatorname{sen}^2 x}{\operatorname{sen}^2 x} \\
 &= -\cot x - x.
 \end{aligned}$$

9. Otro método.

$$\int \cot^2 x = \int (\operatorname{cosec}^2 x - 1) = -\cot x - x.$$

$$\begin{aligned}
 10. \quad \int \operatorname{sen}^2 mx \, dx &= \frac{1}{m} \int (1 - \cos^2 mx) \, dx \\
 &= \frac{1}{4m} (2mx - \operatorname{sen} 2mx)
 \end{aligned}$$

(Perry, 364)

$$11. \quad \int \cos^2 nx \, dx = \frac{1}{n} \int \cos^2 nx \, dx$$

$$= \frac{1}{4n} (2nx + \operatorname{sen}^2 nx)$$

$$12. \int_x \operatorname{sen}^3 x = \frac{1}{4} \int (3 \operatorname{sen} x - \operatorname{sen} 3x)$$

$$= -\frac{1}{4} (3 \cos x + \frac{1}{3} \cos 3x)$$

$$= -\frac{1}{3} \operatorname{sen}^2 x \cos x - \frac{2}{3} \cos x$$

(Peacock, 311)

$$13. \int_x \sec^4 x = \int \sec^2 x \sec^2 x \, dx = \int (1 + \operatorname{tg}^2 x) \, d \operatorname{tg} x$$

$$= \operatorname{tg} x + \frac{1}{3} \operatorname{tg}^3 x. \quad (\text{Roberts, 8})$$

$$14. \int_x \operatorname{sen}^5 x = \frac{1}{16} \int (\operatorname{sen} 5x - 5 \operatorname{sen} 3x + 10 \operatorname{sen} x)$$

$$= \frac{1}{16} \left(-\frac{1}{5} \cos 5x + \frac{5}{3} \cos 3x - 10 \cos x \right)$$

$$15. \int \frac{1}{\operatorname{sen} x} \, dx. \quad (\text{Jarrett, 124})$$

$$\frac{1}{\operatorname{sen} x} = \frac{\operatorname{sen} x}{\operatorname{sen}^2 x} = \frac{\operatorname{sen} x}{1 - \cos^2 x}$$

Descompongamos esta fracción en fracciones parciales:

$$\begin{aligned}\therefore \int \frac{1}{\operatorname{sen} x} dx &= \frac{1}{2} \int \left(\frac{d - \cos x}{1 - \cos x} - \frac{d \cos x}{1 + \cos x} \right) \\ &= \frac{1}{2} L \frac{1 - \cos x}{1 + \cos x} = L \operatorname{tg} \frac{1}{2} x.\end{aligned}$$

$$\begin{aligned}16. \int \frac{1}{\cos x} dx &= \frac{1}{2} \int \left(\frac{d \operatorname{sen} x}{1 + \operatorname{sen} x} - \frac{d - \operatorname{sen} x}{1 - \operatorname{sen} x} \right) \\ &= \frac{1}{2} \left[L(1 + \operatorname{sen} x) - L(1 - \operatorname{sen} x) \right] \\ &= L \sqrt{\frac{1 + \operatorname{sen} x}{1 - \operatorname{sen} x}} = L \cot \left(\frac{1}{4} \pi - \frac{1}{2} x \right) \\ &\quad (\text{Id., id.})\end{aligned}$$

$$\begin{aligned}17. \int \frac{1}{\operatorname{sen} x} dx &= \int \frac{\operatorname{sen}^2 \frac{1}{2} x + \cos^2 \frac{1}{2} x}{4 \operatorname{sen}^2 \frac{1}{2} x \cos^2 \frac{1}{2} x} \\ &= \frac{1}{2} \int \frac{\frac{1}{2} dx}{\cos^2 \frac{1}{2} x} + \frac{1}{2} \int \frac{\frac{1}{2} dx}{\operatorname{sen}^2 \frac{1}{2} x} \\ &= \frac{1}{2} \left(\operatorname{tg} \frac{1}{2} x - \cot \frac{1}{2} x \right) \\ &\quad (\text{De Longchamps, 643})\end{aligned}$$

$$18. \int \frac{dx}{\operatorname{sen} x \cos x} = \int \frac{\operatorname{sen}^2 x \sqrt{\cos^2 x}}{\operatorname{sen} x \cos x} dx$$

$$= \int \frac{\operatorname{sen} x}{\cos x} dx + \int \frac{\cos x}{\operatorname{sen} x} dx$$

$$= -L \cos x + L \operatorname{sen} x = L \operatorname{tg} x.$$

(Timmermans, 252)

$$\begin{aligned} 19. \quad \int \frac{dx}{\operatorname{sen}^2 x \cos^2 x} &= \int \frac{\operatorname{sen}^2 x + \cos^2 x}{\operatorname{sen}^2 x \cos^2 x} dx \\ &= \int \frac{dx}{\cos^2 x} + \int \frac{dx}{\operatorname{sen}^2 x} = \operatorname{tg} x - \cot x \end{aligned}$$

(J. Bertrand, 7)

$$\begin{aligned} 20. \quad \int \frac{\operatorname{sen} x \cos^2 x}{1 + a^2 \cos^2 x} dx &= \frac{1}{a^2} \int \left(\operatorname{sen} x dx - \frac{\operatorname{sen} x dx}{1 + a^2 \cos^2 x} \right) \\ &= -\frac{1}{a^2} \cos x - \frac{1}{a^2} \operatorname{arc} \operatorname{tg} (a \cos x) \end{aligned}$$

$$21. \quad y = \int \frac{1}{a + b \operatorname{tg} x} dx. \text{ Multipliquemos por } \cos x \text{ i-agre-}$$

guemos en seguida

$$\frac{a}{a^2 + b^2} - \frac{a}{a^2 + b^2};$$

$$\frac{1}{a + b \operatorname{tg} x} = \frac{\cos x}{a \cos x + b \operatorname{sen} x} + \frac{a}{a^2 + b^2} - \frac{a}{a^2 + b^2}$$

$$= \frac{a^2 \cos x + b^2 \cos x - a^2 \cos x - a \sin x}{(a \cos x + b \sin x)(a^2 + b^2)} + \frac{a}{a^2 + b^2}$$

$$\therefore y = \frac{b}{a^2 + b^2} \int \frac{d a \cos x + d b \sin x}{a \cos x + b \sin x} + \int \frac{a}{a^2 + b^2} dx$$

$$= \frac{b}{a^2 + b^2} L(a \cos x + b \sin x) + \frac{a x}{a^2 + b^2}$$

$$22. \int \sin p x \cos q x dx$$

$$= \frac{1}{2} \int \sin(p+q)x dx + \frac{1}{2} \int \sin(p-q)x dx$$

$$= -\frac{1}{2} \left[\frac{\cos(p+q)x}{p+q} + \frac{\cos(p-q)x}{p-q} \right]$$

$$22. \int \cos p x \cos q x dx = \frac{1}{2} \left[\frac{\sin(p+q)x}{p+q} + \frac{\sin(p-q)x}{p-q} \right]$$

$$23. \int \cos(ax+b) \cos(a^1x+b^1) dx$$

$$= \frac{\sin[(a+a^1)x+b+b^1]}{a(a+a^1)}$$

$$+ \frac{\sin[(a-a^1)x+b-b^1]}{2(a-a^1)}$$

(Tannery, 511)

$$24. \int \sin(ax+b) \sin(a^1x+b^1) dx$$

$$= \frac{\operatorname{sen} [(a-a^1)x+b-b^1]}{2(a-a^1)} - \frac{\operatorname{sen} [(a+a^1)x+b+b^1]}{2(a+a^1)}$$

(Sturm, 359)

25. $\int \cos x \cos 2x \cos 3x \, dx$. Apliquemos la fórmula

$$\cos a \cos b = \frac{1}{2} \cos (a+b) + \frac{1}{2} \cos (a-b)$$

$$\therefore \cos x \cos 2x = \frac{1}{2} \cos 3x + \frac{1}{2} \cos x$$

$$\therefore \cos x + \cos 2x + \cos 3x = \frac{1}{2} \cos^2 3x + \frac{1}{2} \cos x \cos 3x$$

$$\text{pero } \frac{1}{2} \cos^2 3x = 1 + \cos 6x$$

$$\text{y } \cos x \cos 3x = \frac{1}{2} \cos 4x + \frac{1}{2} \cos^2 x$$

$$\therefore y = \frac{1}{4} \left[\frac{\operatorname{sen} 6x}{6} + \frac{\operatorname{sen} 4x}{4} + \frac{\operatorname{sen} 2x}{2} + x \right]$$

(De Comberousse, 755)

26. $\int \operatorname{sen} x \operatorname{sen} 2x \operatorname{sen} 3x \, dx$.

$$\operatorname{sen} a \operatorname{sen} b = \frac{1}{2} \cos (a-b) - \frac{1}{2} \cos (a+b)$$

$$\therefore \operatorname{sen} x \operatorname{sen} 3x = \frac{1}{2} \cos^2 x - \frac{1}{2} \cos 4x$$

$$\therefore y = \frac{1}{4} \left[x + \frac{\operatorname{sen} 2x}{2} - \frac{\operatorname{sen} 4x}{4} - \frac{\operatorname{sen} 6x}{6} \right]$$

$$27. \int (\operatorname{tg}^2 x - \cot^2 x) dx = \operatorname{tg} x + \cot x = \frac{2}{\operatorname{sen} 2x}.$$

56. *Integración por serie*, o sea desarrollar la derivada en serie i aplicar en seguida la regla de la suma (Núm. 31).

1. $\int (a + bx)^n dx$. Desarrollemos la potencia según el binomio de Newton:

$$\begin{aligned} (a + bx)^n &= a^n + n a^{n-1} b x + \frac{n(n-1)}{2!} a^{n-2} b^2 x^2 \\ &\quad + \frac{n(n-1)(n-2)}{3!} a^{n-3} b^3 x^3 + \dots \end{aligned}$$

Multipliquemos por $x dx$ e integremos cada término:

$$y = \frac{1}{2} a^n x^2 + \frac{1}{3} n a^{n-1} b x^3 + \frac{1}{4} \frac{n(n-1)}{2!} a^{n-2} b^2 x^4 + \dots$$

2. $\int x^m (a + bx)^{\frac{p}{q}} dx$. Es preferible desarrollar así:

$$a + bx^n = a \left(1 + \frac{b}{a} x^n \right)$$

$$\begin{aligned}
 \therefore (a + b x)^{\frac{p}{q}} &= a^{\frac{p}{q}} \left(1 + \frac{b}{a} x \right)^{\frac{p}{q}} \\
 &= a^{\frac{p}{q}} \left[1 + \frac{p}{q} \cdot \frac{b}{a} x + \frac{\frac{p}{q} \left(\frac{p}{q} - 1 \right)}{2!} \frac{b^2}{a^2} x^2 \right. \\
 &\quad \left. + \frac{\frac{p}{q} \left(\frac{p}{q} - 1 \right) \left(\frac{p}{q} - 2 \right)}{3!} \frac{b^3}{a^3} x^3 + \dots \right]
 \end{aligned}$$

Multiplicamos por $x^m dx$ i en seguida integramos:

$$\begin{aligned}
 y &= a^{\frac{p}{q}} \left[\frac{x^{m+1}}{m+1} + \frac{p}{q} \cdot \frac{b}{a} \cdot \frac{x^{m+n+1}}{m+n+1} \right. \\
 &\quad \left. + \frac{\frac{p}{q} \left(\frac{p}{q} - 1 \right)}{2!} \frac{b^2}{a^2} \frac{x^{m+2n+1}}{m+2n+1} + \dots \right]
 \end{aligned}$$

(Timmermans, 385)

3. $\int \frac{1}{1+x} dx$. *I. Método*, por division: efectuemos la di-

vision i encontraremos que

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots \pm x^n \mp \dots$$

Multipiquemos por dx e integremos:

$$\int \frac{dx}{1+x} = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots \pm \frac{x^{n+1}}{n+1} \mp \dots$$

II Método, por el binomio:

$$\begin{aligned} \frac{1}{1+x} &= (1+x)^{-1} = 1 - x + \frac{-1 \cdot -2}{2!} x^2 \\ &\quad + \frac{-1 \cdot -2 \cdot -3}{3!} x^3 - \dots \end{aligned}$$

III Método, por los coeficientes indeterminados:

Supongamos que el desarrollo es,

$$\frac{1}{1+x} = A + Bx + Cx^2 + Dx^3 + Ex^4 + \dots$$

Hagamos desaparecer el denominador:

$$1 = A + Bx + Cx^2 + Dx^3 + Ex^4 + \dots$$

$$1 = \frac{Ax + Bx^2 + Cx^3 + Dx^4 + \dots}{A + (A+B)x + (B+C)x^2 + (C+D)x^3 + \dots}$$

Igualemos dos coeficientes de las mismas potencias de x o sea $x=0$ (Véase el núm. 53):

$$A=1, A+B=0, B+C=0, C+D=0, \dots$$

$$\therefore A=1, B=-1, C=1, D=-1, \dots$$

Sustituyendo, encontramos el mismo desarrollo.

Observación importante. Hemos visto que

$$\int \frac{dx}{1+x} = \int \frac{d(1+x)}{1+x} = L(1+x)$$

luego, igualando con el resultado de mas arriba, llegamos al desarrollo en serie de un logaritmo:

$$L(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 + \dots$$

$$4. \quad \int \frac{1}{a+x} dx = \frac{1}{a}x - \frac{1}{2} \cdot \frac{x^2}{a^2} + \frac{1}{3} \cdot \frac{x^3}{a^3} + \dots$$

(Tedenat, II, 245)

$$5. \quad \int \frac{a dx}{a-x} = ax + \frac{1}{2}x^2 + \frac{1}{3} \cdot \frac{x^3}{a} + \frac{1}{4} \cdot \frac{x^4}{a^2} + \dots$$

$$6. \quad \int \frac{dx}{1+x^2} = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots$$

(Boucharlat, 156)

Por otra parte, sabemos que

$$\int \frac{dx}{1+x^2} = \text{arc. tg } x.$$

Hacemos $\text{arc. tg } x = \frac{1}{4}\pi$, $x=1$, i obtendremos la de

Leibnitz:

$$\frac{1}{4} \pi = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

$$\begin{aligned} 7. \int \frac{a^2}{a^2 + x^2} dx &= \int \left(1 - \frac{x^2}{a^2} + \frac{x^4}{a^4} - \frac{x^6}{a^6} + \dots \right) dx \\ &= x - \frac{1}{3} \frac{x^3}{a^2} + \frac{1}{5} \frac{x^5}{a^4} - \dots \end{aligned}$$

$$\dots a \operatorname{arc} \operatorname{tg} \frac{x}{a} = x - \frac{1}{3} \frac{x^3}{a^2} + \frac{1}{5} \frac{x^5}{a^4} - \dots$$

Esta es la série de Santiago Grégory, (1671)

Haciendo $x = \frac{1}{3} \sqrt{3}$ se obtiene la del Dr. Halley.

(Colin Mac Laurin, 604)

$$8. \int_x \sqrt{1-x^2} = x - \frac{1}{6} x^3 - \frac{1}{40} x^5 - \frac{1}{112} x^7 - \dots$$

(Millar, 37)

$$9. \int_x \sqrt{a^2 + x^2} = a x + \frac{x^3}{6a} + \frac{x^5}{4a^3} + \dots$$

(Stone, 23)

$$10. \int_x \sqrt{a^2 - x^2} = a x - \frac{x^3}{6a} - \frac{x^5}{4a^3} - \frac{x^7}{112a^5} - \dots$$

$$-\frac{5x^9}{1152a^7} - \dots$$

$$11. \int_x \frac{1}{\sqrt{1-x^2}} = x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1.3}{2.4} \frac{x^5}{5} \\ + \frac{1.3.5}{2.4.6} \cdot \frac{x^7}{7} + \dots$$

$$\dots \arcsen x = x + \frac{1}{6} x^3 + \frac{3}{40} x^5 + \frac{1}{164} x^7 + \dots$$

Si hacemos $x=1$ se obtiene la serie de Newton.

(Catalan, 603)

$$12. \int \frac{dx}{\sqrt{1-x^4}} = x + \frac{1}{10} x^5 + \frac{3}{72} x^9 + \frac{15}{624} x^{13} + \dots$$

$$13. \int \frac{a^x}{x} dx. \text{ Desarrollemos la funcion esponencial por}$$

la fórmula de Mac Laurin:

$$a^x = 1 + \frac{xL a}{1!} + \frac{x^2 L^2 a}{2!} + \frac{x^3 L^3 a}{3!} + \dots$$

Multipliquemos por $\frac{dx}{x}$ e integramos:

$$\int \frac{a^x}{x} dx = L x + \frac{L a}{1!} x + \frac{L^2 a}{2!3} x^3 + \frac{L^3 a}{3!4} x^4 + \dots$$

(Pauly, 201)

$$14. \int \frac{e^x}{x} dx = L x + x + \frac{x^2}{2!2} + \frac{x^3}{3!3} + \dots$$

CAPITULO V

INTEGRACION POR SUSTITUCION

57. *Carácter jeneral de este método.*—En la mayor parte de las integrales tratadas hasta aqui, este método está implícitamente empleado o mediante su ayuda se puede hacer la integracion.

En la misma integracion por partes encontramos la sustitucion de la variable.

Sin embargo, no es necesario acudir a este método a cada paso, porque ademas de las dificultades que presenta su correcto empleo, por lo jeneral prolonga los cálculos.

Consiste, como ya se dijo en el número 23, en un *cambio de la variable* o en representar una funcion compuesta por una variable o funcion *auxiliar*, que a su vez puede ser simple o compuesta, monomia, polinomia, inversa, producto, etc.

Por ejemplo, sea

$$dy = f(x) dx.$$

Para eliminar la variable independiente, hacemos

$$x = \phi(z) \quad \therefore \quad dx = \phi'(z) dz,$$

i sustituimos:

$$dy = f[\phi(z)] \phi'(z) dz.$$

La nueva forma ha de ser una integral conocida o de fácil integracion.

58. *Auxiliar monomia.*— Llamamos así a la auxiliar simple z , ax o $a+z$.

$$1. \quad y = \int (a+bx)^3 dx. \quad \text{Sea } a+bx=z \therefore dx = \frac{1}{b} dz.$$

sustituimos:

$$y = \int z^3 \cdot \frac{1}{b} dz = \frac{1}{b} \int z^3 dz = \frac{1}{b} \cdot \frac{1}{4} z^4$$

sustituimos el binomio:

$$y = \frac{1}{b} \cdot \frac{1}{4} (a+bx)^4$$

Se obtiene mas fácilmente la integral introduciendo el coeficiente b .

$$2. \quad y = \int (ax+b)^n dx. \quad ax+b=z \therefore dx = \frac{dz}{a}$$

$$y = \frac{1}{a} \int z^n dz = \frac{1}{a} \cdot \frac{1}{n+1} (ax+b)^{n+1}$$

(Haag, 108)

$$3. \quad y = \int (ax-b)^p dx. \quad ax-b=z \therefore dx = \frac{dz}{a}$$

$$y = \frac{1}{a} \int z^p dz = \frac{1}{a(p+1)} (ax-b)^{p+1}$$

(Timmermans, 247)

$$4. \quad y = \int (a x^n + b)^m = \frac{1}{a n} \int z^{mb} z = \frac{(a x^n + b)^{m+1}}{a n (m+1)}$$

(Lacroix, II, 4)

$$5. \quad y = \int (a + b x^n)^{\frac{p}{q}} x^{m-1} dx.$$

Sea

$$a + b x^n = z \quad \therefore \quad dx = \frac{1}{b n} x^{1-n} dz,$$

$$x^n = \frac{z-a}{b}, \quad x = \left(\frac{z-a}{b} \right)^{\frac{1}{n}}, \quad x^{n-1} = \left(\frac{z-a}{b} \right)^{\frac{1-n}{n}}$$

$$\therefore y = \int z^{\frac{p}{q}} \left(\frac{z-a}{b} \right)^{\frac{m-1}{n}} \cdot \frac{1}{b n} \left(\frac{z-a}{b} \right)^{\frac{1-n}{n}} dz$$

$$= \frac{1}{n b^{\frac{m}{n}}} \int z^{\frac{p}{q}} (z-a)^{\frac{m-n}{n}} dz.$$

(Roberts, 9)

La integracion se termina desarrollando el binomio.

$$6. \quad y = \int \frac{a}{1+a^2 x^2} dx. \quad \text{Sea } x = \frac{v}{a} \quad \therefore \quad dx = \frac{dv}{a}$$

$$y = \int \frac{dv}{1+v^2} = \text{arc tg } a x$$

(Pauly, 158)

$$7. \quad y = \int \frac{dx}{1-(a+x)^2} = \int \frac{dz}{\sqrt{1-z^2}} = \arcsin(a+x)$$

(De Freycinet, 108)

$$8. \quad y = \int \frac{x dx}{a^2+x^2} \quad \text{Sea } a^2+x^2=z \quad \therefore x dx = \frac{1}{2} dz$$

$$y = \frac{1}{2} \int \frac{dz}{z} = \frac{1}{2} L z = L \sqrt{a^2+x^2}$$

(Cournot, II, 2)

$$9. \quad y = \int \frac{dx}{x^2+px+q} \quad \text{Sea } x + \frac{1}{2} p = t \left(q - \frac{1}{4} p^2 \right)^{\frac{1}{2}}$$

$$\therefore dx = dt \sqrt{q - \frac{1}{4} p^2}$$

$$y = \int \frac{\left(q - \frac{1}{4} p^2 \right)^{\frac{1}{2}}}{\left(q - \frac{1}{4} p^2 \right) t^2 + \left(q - \frac{1}{4} p^2 \right)} dt$$

$$= \frac{1}{\sqrt{q - \frac{1}{4} p^2}} \int \frac{dt}{1+t^2} = \frac{1}{\sqrt{q - \frac{1}{4} p^2}} \arctg \frac{x + \frac{1}{2} p}{\sqrt{q - \frac{1}{4} p^2}}$$

(Serret, II, 16)

$$10. \quad \int \frac{dx}{a+2bx+cx^2} = \int \frac{cdx}{(cx+b)^2+ac-b^2}$$

Sea

$$c x + b = z \quad \therefore \quad d x = \frac{1}{c} d z$$

$$y = \int \frac{d z}{a c - b^2 + z^2} = \frac{1}{\sqrt{a c - b^2}} \operatorname{arc} \operatorname{tg} \frac{c x + b}{\sqrt{a c - b^2}}$$

$$o \quad y = \frac{1}{\sqrt{b^2 - a c}} L \frac{c x + b - \sqrt{b^2 - a c}}{c x + b + \sqrt{b^2 - a c}}$$

(Roberts, 11)

$$11. \quad y = \int \frac{d x}{(x-a)(x-b)} \quad \text{Sea } x = \frac{1}{2}(a+b) + z \quad \therefore \quad d x = d z,$$

$$x - a = z + \frac{1}{2}(b-a)$$

$$x - b = z - \frac{1}{2}(b-a)$$

$$\therefore (x-a)(x-b) = z^2 - \frac{1}{4}(b-a)^2$$

$$y = \int \frac{d z}{z^2 - \frac{1}{4}(b-a)^2} = \frac{1}{a-b} L \frac{x-a}{x-b}$$

$$12. \quad y = \int \frac{5 x^3}{3^4 x + 7} \quad \text{Sea } 3 x^4 + 7 = t \quad \therefore \quad x^3 d x = \frac{1}{12} d t$$

$$y = \frac{5}{12} \int \frac{d t}{t} = \frac{5}{12} (3 x^4 + 7)$$

(Sturm, 317)

$$\begin{aligned}
 13. \quad y &= \int \frac{dx}{x(a+bx^m)} = \int \frac{x^{m-1} dx}{x^m(a+bx^m)} \\
 &= \frac{1}{m} \int \frac{dz}{z(a+bz)} = \frac{1}{am} L \frac{x^m}{a+bx^m} \\
 &\quad (\text{Jordan, II, 11})
 \end{aligned}$$

$$\begin{aligned}
 14. \quad y &= \int \frac{x dx}{(a+bx)^3} \cdot \text{Sea } a+bx=z \dots dx = \frac{1}{b} dz \\
 y &= \frac{1}{b^2} \int (z^{-2} - az^{-3}) dz = \frac{-1}{2b^2} \cdot \frac{2bx+a}{(a+bx)^{\frac{2}{2}}}
 \end{aligned}$$

$$\begin{aligned}
 15. \quad y &= \frac{x dx}{(a+bx)^{\frac{2}{3}}} = \frac{1}{b^2} \int \frac{z-a}{z^{\frac{2}{3}}} dz \\
 &= \frac{3}{4} \cdot \frac{1}{b^2} (bx-3a)(a+bx)^{\frac{1}{3}}
 \end{aligned}$$

$$\begin{aligned}
 16. \quad y &= \int \frac{x dx}{(x^2+a^2)^n} = \frac{1}{2} \int z^{-n} dz = -\frac{1}{2(n-1)(x^2+a^2)^{n-1}} \\
 &\quad (\text{Brahya, 14})
 \end{aligned}$$

$$\begin{aligned}
 17. \quad y &= \int \frac{Ax^m}{(a+bx)^n} dx = \frac{A}{b} \int \left(\frac{z-a}{b} \right) \frac{1}{z^n} dz \\
 &= \frac{A}{b^{m+1}} \int \left(\frac{z-a}{z^n} \right)^m dz
 \end{aligned}$$

Desarrollamos por el binomio e integramos cada término:

$$y = \frac{A}{b^{m+1}} \left[\frac{(a+bx)^{m-n+1}}{m-n+1} - \frac{ma(a+bx)^{m-n}}{m-n} + \frac{a^2m(m-1)(a+bx)^{m-n-1}}{m-n-1} + \dots \right]$$

18. $y = \int \frac{dx}{(a+bx^n)^{\frac{n+1}{n}}}$ Multipliquemos los dos térmi-

nos por x^{-n-1} i en seguida hacemos a $x^{-n}+b=z$

$$y = \frac{1}{a} \frac{x}{(a+bx^n)^{\frac{1}{n}}}$$

19. $y = \int \sqrt{ax+b} \cdot dx$ Sea $dx+b=z$ $dx = \frac{1}{a} dz$

$$y = \frac{1}{a} \int z^{\frac{1}{2}} dz = \frac{2}{3a} \sqrt{(ax+b)^3}$$

20. $y = \int \sqrt{a^2-x^2} \cdot x dx = -\frac{1}{2} \int z^{\frac{1}{2}} dz$

$$= -\frac{1}{3} (a^2-x^2)^{\frac{3}{2}}$$

(De Morgan, 107)

21. $y = \int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{a dx}{(a^2-a^2 z^2)^{\frac{1}{2}}} = \int \frac{dz}{(1-z^2)^{\frac{1}{2}}}$

$$= \text{arc sen. } \frac{x}{a} \quad (\text{Pruvost, I, 550})$$

$$22. \quad y = \int \frac{x \, dx}{\sqrt{1-x^2}} \quad \text{Sea } 1-x^2=z \quad \dots \quad x \, dx = -\frac{1}{2} dz$$

$$y = -\frac{1}{2} \int z^{-\frac{1}{2}} \, dz = -\sqrt{1-x^2}$$

$$23. \quad y = \int \frac{dx}{\sqrt{2ax-x^2}} \quad \text{Sea } x=a-z \quad \dots \quad dx = -dz$$

$$y = \int \frac{-dz}{\sqrt{a^2-z^2}} = \arccos \frac{a-x}{a} = \operatorname{arcvers} \frac{x}{a}$$

$$24. \quad y = \int \frac{dx}{\sqrt{7+6x-x^2}} = \int \frac{dx}{\sqrt{16-(x-3)^2}}$$

Sea

$$x=3+4u \quad \dots \quad dx=4 \, du$$

$$y = \int \frac{du}{\sqrt{1-u^2}} = \arcsen \frac{x-3}{4} \quad (\text{Sonnet, 217})$$

$$25. \quad y = \int \frac{dx}{\sqrt{a+bx-x^2}} = \int \frac{dx}{\sqrt{a+\frac{b^2}{4}-\left(x-\frac{b}{2}\right)^2}}$$

Sea

$$\frac{1}{2}b-x=t \sqrt{a+\frac{1}{4}b^2} \quad \dots \quad dx = -\sqrt{a+\frac{1}{4}b^2} \, dt$$

$$y = - \int \frac{dt}{\sqrt{1-t^2}} = \arccos \frac{b-2x}{\sqrt{4a+b^2}}$$

(Sturm, 1340)

$$26. \quad y = \int \frac{dx}{\sqrt{a+bx+cx^2}} = \frac{1}{\sqrt{c}} \int \frac{dx}{\sqrt{\left(x + \frac{b}{2c}\right)^2 + \frac{4ac-b^2}{4c^2}}}$$

Sea

$$x + \frac{b}{2c} = z$$

$$dx = dz$$

$$y = \frac{1}{\sqrt{c}} L \left[2cx+b + 2\sqrt{c} \sqrt{a+bx+6x^2} \right]$$

(Todhunter, 12)

$$27. \quad y = \int \frac{3x-2}{\sqrt{4x^2+16x+20}} dx = \int \frac{3x-2}{\sqrt{4[(x-2)^2+1]}} dx$$

Sea

$$x=2+u \quad dx=du$$

$$y = \frac{1}{2} \int \frac{u du}{\sqrt{u^2+1}} + \int \frac{du}{\sqrt{u^2+1}}$$

$$= 2L[(x-2) + \sqrt{x^2-4x+5}] + \frac{3}{2} \sqrt{x^2-4x+5}$$

$$28. \quad y = \int e^{ax} dx \quad \text{Sea } ax=z \quad dx = \frac{1}{a} dz$$

$$y = \int e^z \cdot \frac{1}{a} dz = \frac{1}{a} e^{ax}$$

$$29. \quad y = \int e^{-ax} dx = -\frac{1}{a} \int e^z dz = -\frac{1}{a} e^{-ax}$$

$$30. \quad \int e^{f(x)} df(x) = \int e^z dz = e^{f(x)}$$

(Moigno, II, 12)

$$31. \quad y = \int \frac{e^x x dx}{(1+x)^2} \quad \text{Sea } 1+x=z \quad \therefore dx=dz$$

$$y = \int \frac{e^z}{e} \left(\frac{dz}{z} - \frac{dz}{z^2} \right) = \frac{e^z}{ez} = \frac{e^x}{1+x}$$

(Francoeur, 374)

$$32. \quad \int \frac{dx}{1+e^x} \quad \text{Sea } 1+e^x=z \quad \therefore e^x dx=dz,$$

$$dx = \frac{dz}{z^2} = \frac{dz}{z}$$

$$y = \int \frac{dz}{z^2} = -z^{-1} = \frac{-1}{1+e^x}$$

$$33. \quad \int \frac{dx}{e^x + e^{-x}} \quad \text{Sea } e^x = y \quad \therefore dx = e^{-x} dy = \frac{dz}{z}$$

$$y = \int \frac{dz}{z(z+z^{-1})} = \int \frac{dz}{z^2+1} = \text{arc tg } e^x$$

(Byerly, II, 36)

34. $\int f(e^{ax}) dx$. Sea $e^{ax} = \theta \therefore ax = L\theta$,

$$dx = \frac{d\theta}{a\theta}, y = \frac{1}{a} \int f(\theta) \frac{d\theta}{\theta}$$

(Hermite, I, 255)

35. $\int \frac{Lx}{x} dx$. Sea $Lx = z \therefore \frac{dx}{x} = dz$

$$y = \int z dz = L^2 \sqrt{x}$$

(Niewenglowski, II, 145)

36. $y = \int \frac{dx}{x L^n x}$. Sea $Lx = z \therefore \frac{dx}{x} = dz$

$$y = \frac{dz}{z^n} = \frac{1}{1-n} (Lx)^{1-n}$$

37. $y = \int \sqrt{1+Lx} \frac{dx}{x}$. Sea $1+Lx = z \therefore \frac{dx}{x} = dz$

$$y = \int \sqrt{z} dz = \frac{2}{3} (1+Lx)^{\frac{3}{2}}$$

(Bertrand, 8)

38. $y = \int \frac{dx}{x} f(Lx) = \int f(z) dz = f L(x)$.

39. $y = \int \operatorname{sen} ax dx$. Sea $ax = z \therefore dx = \frac{1}{a} dz$

$$y = \frac{1}{a} \int \operatorname{sen} z \, dz = -\frac{1}{a} \cos a x.$$

40. $y = \int \frac{dx}{\operatorname{sen} x}$ Sea $\cos x = z$

$$\begin{cases} 1 - \cos^2 x = 1 - z^2, & \operatorname{sen} x = \sqrt{1 - z^2} \\ -\operatorname{sen} x \, dx = dz & dx = -\frac{dz}{\operatorname{sen} x} \end{cases}$$

$$y = -\int \frac{dz}{1 - z^2} = L \sqrt{\frac{1 - \cos x}{1 + \cos x}} = L \operatorname{tg} \frac{1}{2} x$$

(Francoeur, 381)

41. $y = \int \cos x \, dx$. Sea $\operatorname{sen} x = z \therefore \cos x \, dx = dz$

$$\therefore y = \int dz = z = \operatorname{sen} x. \quad (\text{Roberts, 7})$$

42. $y = \int \frac{dx}{\cos x} \quad dx = \int \frac{\cos x \, dx}{\cos^2 x}$

Sea

$$z = \operatorname{sen} x \quad \therefore 1 - z^2 = \cos^2 x,$$

$$dz = \cos x \, dx, \quad y = \int \frac{dz}{1 - z^2} = L \sqrt{\frac{1 + z}{1 - z}}$$

$$= L \sqrt{\frac{1 + \operatorname{sen} x}{1 - \operatorname{sen} x}} = L \cot \left(\frac{1}{4} \pi - \frac{1}{2} x \right)$$

43. $y = \int \operatorname{sen} x \cos x \, dx$. Sea $\operatorname{sen} x = z \therefore \cos x \, dx = dz$

$$y = \int z \, dz = \frac{1}{2} \operatorname{sen}^2 x$$

44. $\int \operatorname{sen}^m x \cos^p x \, dx$. Sea $p=2k+1$, $\operatorname{sen} x = u$,

$$\cos x = (1-u^2)^{\frac{1}{2}}, \, dx = (1-u^2)^{-\frac{1}{2}} du$$

$$y = \int u^m (1-u^2)^k du.$$

Sea

$$m=6, \, p=9$$

$$\int \operatorname{sen}^6 x \cos^9 x \, dx = \int u^6 (1-u^2)^4 du.$$

$$= \frac{1}{7} \operatorname{sen}^7 x - \frac{4}{9} \operatorname{sen}^9 x + \frac{6}{11} \operatorname{sen}^{11} x - \frac{4}{13} \operatorname{sen}^{13} x$$

$$+ \frac{1}{15} \operatorname{sen}^{15} x. \quad (\text{Sonnet, 232})$$

45. $y = \int \sec^3 x \, dx$. Sabemos que la $\sec$. es la inversa del $\cos$:

$$\sec^3 x = \frac{1}{\cos^3 x} = \frac{\cos x}{\cos^4 x}, \, \cos x = d \operatorname{sen} x:$$

$$y = \int \frac{d \operatorname{sen} x}{(1-\operatorname{sen}^2 x)^2}. \quad \text{Hacemos } \operatorname{sen} x = t \text{ y queda}$$

$$y = \frac{dt}{(1-t^2)^2}$$

Integral que se calculó en el ejercicio 27 del número 53:

$$y = \frac{1}{4} \left[L \frac{1+t}{1-t} + \frac{2t}{1-t^2} \right]$$

Reemplazamos el valor de $t = \sin x$:

$$y = \frac{1}{4} \left[L \frac{1+\sin x}{1-\sin x} + 2 \operatorname{tg} x \sec x \right]$$

Siendo

$$\frac{1+\sin x}{1-\sin x} = \frac{(1+\sin x)^2}{1-\sin^2 x} = \frac{(1+\sin x)^2}{\cos^2 x} = (\sec x + \operatorname{tg} x)^2$$

encontraremos que

$$y = \frac{1}{2} (\sec x + \operatorname{tg} x) + \frac{1}{2} \operatorname{tg} x \sec x$$

En función de la tangente será:

$$y = \frac{1}{2} L(\operatorname{tg} x + \sqrt{\operatorname{tg}^2 x + 1}) + \frac{1}{2} \operatorname{tg} x \sqrt{\operatorname{tg}^2 x + 1}$$

$$45. \quad y = \int \frac{dx}{1+a \cos x + b \sin x} \quad \text{Sea } x = y - \operatorname{arc} \operatorname{tg} \frac{a}{b}$$

$$\text{y sea } \operatorname{arc} \operatorname{tg} \frac{a}{b} = m$$

$$\therefore \operatorname{tg} m = \frac{a}{b}, \cos m = \frac{b}{\sqrt{a^2 + b^2}}, \sin m = \frac{a}{\sqrt{a^2 + b^2}}$$

$$a \cos x = a \cos y \cos m + a \sin y \sin m$$

$$\frac{b \sin x = b \sin y \cos m - b \sin m \cos y}{a \cos x + b \sin x = \sin y (a \sin m + b \cos m)}$$

$$+ \cos y (a \cos m - b \sin m)$$

$$= \sin y \left(\frac{a b}{\sqrt{a^2 + b^2}} + \frac{b^2}{\sqrt{a^2 + b^2}} \right)$$

$$+ \cos y \left(\frac{a b}{\sqrt{a^2 + b^2}} - \frac{a b}{\sqrt{a^2 + b^2}} \right)$$

$$= \sqrt{a^2 + b^2} \sin y$$

$$\therefore y = \int \frac{d y}{1 + \sqrt{a^2 + b^2} \sin y}, \text{ forma ya tratada.}$$

$$46. \int \frac{d x}{\operatorname{tg} x - \operatorname{tg} a} \quad \text{Sea } \operatorname{tg} x = t \quad \sec^2 x \, d x = d t,$$

$$1 + \operatorname{tg}^2 x = 1 + t^2 \quad d x = \frac{d t}{1 + t^2}$$

$$y = \int \frac{d t}{(1 + t^2)(t - \operatorname{tg} a)} = \cos^2 a \int \frac{d t}{t - \operatorname{tg} a} - \cos^2 a \int \frac{t \, d t}{1 + t^2}$$

$$- \sin a \cos a \int \frac{d t}{1 + t^2}$$

$$= \cos^2 a \, L \frac{\operatorname{tg} x - \operatorname{tg} a}{\sqrt{1 + \operatorname{tg}^2 a}} - x \sin a \cos a$$

59. *Auxiliar inversa*, o sea sustituir x por $\frac{1}{z}$ o $\frac{1}{x}$ por z .

$$1. \int \frac{x dx}{(a+bx)^3} \cdot \text{Sea } a+bx = \frac{1}{z} \therefore dx = -\frac{1}{bz^2} dz;$$

uego,

$$x = \frac{1-az}{bz}, \quad x dx = \frac{az-1}{b^2 z^3} dz$$

$$\int \frac{x dx}{(a+bx)^3} = \frac{1}{b^2} \int (az-1) dz = \frac{1}{b^2} \left(\frac{1}{2} az^2 - z \right)$$

$$= \frac{a - \frac{2}{z}}{2b^2} \cdot z^2 = -\left(\frac{1}{2b^2} + \frac{x}{b} \right) \frac{1}{(a+bx)^2}$$

(Peacock, 269)

$$2. \int \frac{dx}{x \sqrt{x^2-1}} \cdot \text{Sea } x = \frac{1}{u} \therefore dx = -\frac{du}{u^2};$$

luego,

$$\frac{dx}{u} = -\frac{du}{u};$$

queda:

$$\int \frac{dx}{x \sqrt{x^2-1}} = \int \frac{-du}{u \sqrt{\frac{1}{u^2}-1}} = \int \frac{-du}{\sqrt{1-u^2}} = \arccos \frac{1}{x}$$

$$3. \int \frac{dx}{x(a+bx^2)} \quad \text{Sea } x^2 = \frac{1}{z} \quad \therefore \quad \frac{dx}{x} = -\frac{dz}{2z};$$

luego

$$y = -\frac{1}{2a} \int \frac{adz}{az+b} = -\frac{1}{2a} L(az+b)$$

$$= \frac{1}{a} L \sqrt{\frac{x^2}{a+bx^2}}$$

$$4. \int \frac{dx}{x \sqrt{x^2-a^2}} \quad \text{Sea } x = \frac{1}{y} \quad \therefore \quad \frac{dx}{x} = -\frac{dy}{y}$$

$$= \int -\frac{dy}{y \sqrt{\frac{1}{y^2}-a^2}} = -\int \frac{dy}{\sqrt{1-(ay)^2}} = -\frac{1}{a} \arcsen ax.$$

(Todhunter, II, 13)

$$5. \int \frac{dy}{x \sqrt{a^2 \pm x^2}} \quad \text{Sea } x = \frac{1}{y}$$

$$= \frac{1}{a} L \frac{x}{a + \sqrt{a^2 \pm x^2}}$$

$$6. \int \frac{dx}{x \sqrt{2ax-a^2}} \quad \text{Sea } x = \frac{1}{1-z} \quad \therefore \quad \frac{dx}{x} = \frac{adz}{1-z}$$

$$\int \frac{adz}{(1-z) \sqrt{\frac{2a^2}{1-z}-a^2}} = \frac{1}{a} \int \frac{dz}{\sqrt{1-z^2}} = \frac{1}{a} \arcsen \frac{x-a}{x}$$

$$7. \int \frac{dx}{\sqrt{4x^2 - 4x - 1}} \quad \text{Sea } x = \frac{1}{z} \quad \therefore \frac{dx}{x} = -\frac{dz}{z}$$

$$\therefore y = \int \frac{-dz}{\sqrt{4 - 4z - z^2}} = \int \frac{-dz}{\sqrt{8 - (z+2)^2}} = \arccos \frac{2x+1}{2\sqrt{2}x}$$

(Brahya, 14)

$$8. \int \frac{dx}{(1-x^2)^{\frac{3}{2}}} \quad \text{Traspongamos la variable:}$$

$$\frac{1}{(1-x^2)^{\frac{3}{2}}} = \frac{1}{\left[x^2(x^{-2}-1)\right]^{\frac{3}{2}}} = \frac{x^{-3}}{(x^{-2}-1)^{\frac{3}{2}}}$$

$$= -\frac{1}{2} \cdot \frac{-2x^{-3}}{(x^{-2}-1)^{\frac{2}{2}}}$$

Sea ahora,

$$x^{-2}-1=z \quad \therefore -2x^{-3}dx = dz$$

$$y = -\frac{1}{2} \int z^{-\frac{3}{2}} dz = -\frac{x}{\sqrt{1-x^2}}$$

$$9. \frac{du}{dx} = \frac{1}{x^m(u+bx^n)} \quad \text{Sea } x = \frac{1}{z} \quad \therefore \frac{dx}{dz} = -\frac{1}{z^2}$$

$$\frac{du}{dz} = \frac{du}{dx} \cdot \frac{dx}{dz} = -\frac{1}{z^2} \cdot \frac{du}{dx} = -\frac{z^{m+n-2}}{(az+b)^n}$$

$$u = - \int \frac{z^{m+n-2}}{z(a+z)^n} dz$$

hacemos ahora

$$a z + b = t \quad \therefore a dz = dt, \quad z = \frac{t-b}{a}$$

$$u = - \int \left(\frac{t-b}{a} \right)^{m+n-2} \cdot \frac{dt}{a t^n}$$

$$= - \frac{1}{a^{m+n-1}} \int \frac{(t-b)^{m+n-2}}{t^n} dt$$

Desarrollamos por el binomio e integramos cada término.

(Hall, 222)

$$10. \int \frac{dx}{x \sqrt{a^2 \pm x^2}} \quad \text{Sea } x = \frac{1}{z} \text{ i sea } a = \frac{1}{m}$$

$$y = - \int \frac{dz}{\sqrt{z^2 \pm m^2}} = - L \frac{a \pm \sqrt{a^2 \pm x^2}}{a x}$$

(Cox, 34)

60. *Auxiliar producto.*

$$1. \int \frac{dx}{x(a+bx)^2} \quad \text{Sea } a+bx = zx \quad dx = - \frac{a}{(z-b)^2} dz$$

$$y = - \frac{1}{a^2} \int \frac{z-b}{z^3} dz = - \frac{1}{a^2} \left[L(a+bx) + \frac{b}{a+bx} \right]$$

$$2. \int \frac{dx}{x^m(a+bx)^n} \quad \text{Sea } a+bx=z \quad \therefore x = \frac{a}{z-b},$$

$$dx = -\frac{a}{(z-b)^2} dz,$$

$$y = \frac{1}{a^{m+n-1}} \int \frac{(z-b)^{m+n-2}}{z^n} dz$$

Es integrable cuando sea $m+n > 1$

(Williamson, II, 6)

$$3. y = \int \sqrt{a^2 - x^2} dx \quad \text{Sea } \sqrt{a^2 - x^2} = (a-x)z$$

$$dy = \frac{8a^2 z^2 dz}{(1+z^2)^3} = -\frac{8a^2 dz}{(1+z^2)^3} + \frac{8a dz}{(1+z^2)^2}$$

$$y = \frac{1}{2} x \sqrt{a^2 - x^2} + a^2 \arctan \left(\sqrt{\frac{a+x}{a-x}} \right)$$

(Fraucoeur, 367)

$$4. \int \frac{dx}{\sqrt{1-x^2}} \quad \text{Sea } \sqrt{1-x^2} = (1-x)z$$

$$1-x^2 = z^2 - 2xz + x^2 z^2$$

$$x^2(1+z^2) - 2xz + z^2 - 1 = 0$$

$$x^2 - \frac{2z^2}{1+z^2} x + \frac{z^2-1}{z^2+1} = 0$$

$$x = \frac{z^2 - 1}{z^2 + 1}, \quad dx = \frac{4z dz}{(z^2 + 1)^2}$$

$$\int \frac{dx}{\sqrt{1-x^2}} = 2 \int \frac{dz}{1+z^2} = 2 \operatorname{arc} \operatorname{tg} \sqrt{\frac{1+x}{1-x}} = \operatorname{arc} \operatorname{sen} x.$$

5. $\int \frac{dx}{\sqrt{a+bx+x^2}}$ Sean a, β las raíces reales:

$$\sqrt{a+bx+x^2} = \sqrt{(x-a)(x-\beta)} = (x-a)z$$

$$\beta - x = (x-a)z^2 \quad \therefore x = \frac{\beta + az^3}{1+z^2}$$

$$dx = \frac{2(a-\beta)z}{(1+z^2)^2} dz, \quad y = -2 \int \frac{dz}{1+z^2}$$

$$= -2 \operatorname{arc} \operatorname{tg} \sqrt{\frac{\beta-x}{x-a}}$$

6. $\int \frac{dx}{x \sqrt{1-x^2}}$ Sea $\sqrt{1-x^2} = xz-1$,

$$\therefore 1-x^2 = x^2 z^2 - 2xz + 1 \quad \text{o} \quad x = \frac{2z}{1+z^2},$$

$$xz = \frac{2z^2}{1+z^2}, \quad \sqrt{1-x^2} = \frac{z^2-1}{z^2+1}, \quad dx = \frac{2+2z^2-4z^2}{1+z^2}$$

$$\therefore y = - \int \frac{dz}{z} = -L \frac{x}{1+\sqrt{1-x^2}}$$

$$7. \int \frac{dx}{x \sqrt{1 \pm x^2}} = L \frac{x}{1 + \sqrt{1 \pm x^2}}$$

$$8. \int \frac{dx}{x \sqrt{a^2 \pm x^2}} = \frac{1}{a} L \frac{x}{a + \sqrt{a^2 \pm x^2}}$$

(Brady, 17)

$$9. \int \frac{1+x^2}{(1-x^2) \sqrt{1+x^4}} \cdot \text{Sea } \frac{x \sqrt{2}}{1+x^2} = u$$

$$y = \frac{1}{\sqrt{2}} L \frac{\sqrt{1+x^4} + \sqrt{2}}{1-x^4}$$

(Euler, IV, 22)

$$10. \int \frac{\sqrt{1+x^4}}{1-x^4} dx.$$

Sea

$$u = \frac{x \sqrt{2}}{\sqrt{1+x^4}}$$

$$y = \frac{1}{2\sqrt{2}} L \frac{\sqrt{1+x^4} + x \sqrt{2}}{1-x^2} + \frac{1}{2\sqrt{2}} \arcsen \frac{x \sqrt{2}}{1+x^2}$$

61. *Auxiliar potencia.*

$$1. y = \int x \sqrt{a-x} dx. \text{ Sea } a-x=z^2. \quad x=a-z^2,$$

$$dx = -2z dz, \quad y = -\int (a-z^2)^2 z^2 dz$$

$$= \frac{2}{3} a (a-x)^{\frac{3}{2}} - \frac{2}{5} (a-x)^{\frac{5}{2}}.$$

(Roberts, 10)

2. $y = \int x^2 \sqrt{a+x} dx$. Sea $a+x = t^2 \dots dx = 2t dt$

$$y = 2 \int (t^2 - a)^2 t^2 dt = 2 \left(\frac{t^7}{7} - \frac{2at^5}{5} + \frac{a^2 t^3}{3} \right)$$

$$= 2(a+x)^{\frac{3}{2}} \left\{ \frac{(a+x)^2}{7} - \frac{2a}{5}(a+x) + \frac{a^2}{3} \right\}$$

(Todhunter, 18)

3. $y = \int \frac{dx}{\sqrt{1+x^2}}$. Hacemos $1+x^2 = z^2$

Diferenciamos:

$$2x dx = 2z dz \dots \frac{dx}{z} = \frac{dz}{x}$$

Componemos:

$$\frac{dx + dz}{z+x} = \frac{d(x+z)}{z+x} = \frac{dx}{z}$$

$$\therefore y = \int \frac{dx}{z} = \frac{d(z+x)}{z+x} = L(z+x)$$

$$\therefore \int \frac{dx}{\sqrt{1-x^2}} = L(x + \sqrt{1+x})$$

$$4. \int \frac{x dx}{\sqrt{1-x^2}} \quad \text{Sea } 1-x^2=z^2 \quad \therefore x dx = -z dz.$$

$$y = - \int dz = -z = -\sqrt{1-x^2}$$

$$5. \int \frac{dx}{\sqrt{a^2-x^2}} \quad \text{Sea } x^2+a^2=z^2 \quad \therefore \frac{dx}{z} = \frac{dz}{x},$$

$$\frac{d(x+z)}{x+z} + \frac{dx}{z}$$

$$y = \int \frac{d(x+z)}{x+z} = L(x + \sqrt{a^2+x^2})$$

(Osborne, 185).

$$5. \int \frac{x dx}{\sqrt{a^2+x^2}} \quad \text{Sea } a^2+x^2=t^2 \quad \therefore x dx = t dt$$

$$y = \sqrt{a^2+x^2}$$

$$7. \int \frac{\sqrt{x}+1}{2x\sqrt{x}} dx \quad \text{Sea } x=z^2 \quad \therefore dx=2z dz$$

$$y = \int \frac{z+1}{z} dz = \int (z^{-1} + z^{-2}) dz$$

$$= Lz - \frac{1}{z} = L\sqrt{x} - \frac{1}{\sqrt{x}}$$

$$8. \int \frac{dx}{\sqrt{x^2 \pm 1}} \cdot \text{Sea } x^2 \pm 1 = u^2$$

$$2x dx = 2u du$$

$$\therefore \frac{d(x+u)}{x+u} = \frac{dx}{\sqrt{x^2 \pm 1}}$$

$$\int \frac{dx}{\sqrt{x^2 \pm 1}} = L(x + \sqrt{x^2 \pm 1})$$

(Jarrett, 123)

$$9. \int \frac{dx}{x^2 \sqrt{1+x^2}} \cdot \text{Sea } x^2 + 1 = t^2$$

$$y = -\frac{\sqrt{x^2 + 1}}{x}$$

$$10. \int \frac{dx}{(a^2 + x^2)^{\frac{3}{2}}} = \int \frac{\frac{dx}{dt}}{x^3(a^2 x^{-2} + 1)^{\frac{3}{2}}} dt = -\frac{1}{a^2} \int \frac{dt}{t^2}$$

$$= \frac{x}{a^2 \sqrt{a^2 + x^2}}$$

$$11. \int \frac{dx}{\sqrt{(x-a)(x-\beta)}} \cdot \text{Sea } x-a = z$$

$$\frac{dx}{\sqrt{x-a}} = 2 dz$$

$$y = \int -\frac{2 dz}{\sqrt{z^2 + \alpha - \beta}}$$

$$= 2 \log (\sqrt{x - \alpha} + \sqrt{x - \beta})$$

12. $\int \frac{x dx}{\sqrt{a^4 - x^4}}$. Sea $x^4 = a^4 z^2$, $x dx = \frac{1}{2} a^2 dz$

$$y = \frac{1}{2} \arcsen \frac{x^2}{a^2}$$

(Raffy, 84)

62. Auxiliar binomia.

1. $y = \int \sqrt{a^2 + x^2} dx$. Sea $\sqrt{a^2 + x^2} = z - x$

$$y = \int (z - x) dx = -\frac{1}{2} x^2 + \int z dx$$

Para eliminar la dx , elevamos al cuadrado

$$a^2 + x^2 = (z - x)^2 \quad x = \frac{z^2 - a^2}{2z}$$

$$dx = \frac{z^2 + a^2}{2z^2} dz$$

$$y = -\frac{1}{2} x^2 + \int \frac{z^2 + a^2}{2z} dz$$

$$= -\frac{1}{2} x^2 + \frac{1}{2} \int z dz + \frac{a^2}{2} \int \frac{dz}{z}$$

$$= \frac{1}{2} x \sqrt{a^2 + x^2} + \frac{a^2}{2} L(x + \sqrt{a^2 + x^2})$$

(Francoeur, 366)

2. $dy = \sqrt{c + bx + x^2} dx$. Representemos el radical por $z + x$ i elevemos al cuadrado:

$$bx + c = z^2 + 2xz \dots x = \frac{z^2 - c}{b - 2z} \dots dx = 2 \frac{bz - z^2 - c}{(b - 2z)^2} dz$$

$$y = \int \left(z + \frac{z^2 - c}{b - 2z} \right) \cdot 2 \frac{bz - z^2 - c}{(b - 2z)^2} dz$$

$$= 2 \int \frac{bz - z^2 - c}{(b - 2z)^3} dz. \quad (\text{Timmermans, 272})$$

La integracion se puede hacer por division.

3. $\int \frac{dx}{\sqrt{1+x^2}}$ Sea $\sqrt{1+x^2} = v - x$

$$v^2 - 2xv = 1$$

$$\int \frac{dx}{\sqrt{1+x^2}} = \int \frac{dx}{v-x} = \int \frac{dv}{v}$$

$$= L(x + \sqrt{1+x^2}) \quad (\text{Pauly, 173})$$

4. $\int \frac{dx}{\sqrt{x^2+a^2}}$ Sea $\sqrt{x^2+a^2} = z - x \therefore a^2 = z^2 - 2xz$,

$$\frac{dx}{\sqrt{x^2+a^2}} = \frac{dx}{z-x} = dLz$$

$$y = L(x + \sqrt{x^2+a^2})$$

(Todhunter, II, 9)

$$5. \int \frac{dx}{\sqrt{a^2+x^2}} \quad \text{Sea } \sqrt{a^2+x^2} = x+z \therefore dx - \frac{z^2+a^2}{2z^2} dz$$

$$y = - \int \frac{dz}{z} = L(x + \sqrt{x^2+a^2})$$

(Laurent, III, 41)

$$6. \int \frac{dx}{\sqrt{x^2 \pm a^2}} = L(x + \sqrt{x^2 \pm a^2})$$

$$7. \int \frac{dx}{\sqrt{2ax-x^2}} \quad x=a-z \therefore \frac{dx}{dz} = -1$$

$$y = - \int \frac{dz}{\sqrt{a^2-z^2}} = \text{arc cos} \frac{a-x}{a}$$

$$8. \int \frac{m a^m x^{m-1}}{\sqrt{(ax)^{2m} \pm b^{2n}}} dx = L \frac{(ax)^m + \sqrt{(ax)^{2m} + b^{2n}}}{b^n}$$

(Brahya, 19)

$$9. \int \frac{dx}{\sqrt{Ax^2+Bx+C}} \quad \text{Se representa el radical por}$$

$z-x\sqrt{A}$ i se eleva al cuadrado:

$$B dx = 2z dz - 2z A dx - 2x \sqrt{A} dz:$$

$$y = \frac{1}{\sqrt{a}} L \left(\frac{1}{2} B + z \sqrt{a} \right) = \frac{1}{\sqrt{a}} L \left(Ax + \frac{B}{2} + \sqrt{A} \sqrt{Ax^2 + Bx + C} \right)$$

(Moigno, II, 20)

63. *Sustitucion trigonométrica.*

1. $\int \sqrt{1-x^2} dx$. Sea $x = \sin \theta$

$$\therefore dx = \cos \theta d\theta, \cos \theta = \sqrt{1-x^2}$$

$$\therefore y = \int \cos^2 \theta d\theta = \int \frac{1 + \cos 2\theta}{2} d\theta$$

Pero, de $\sin \theta = x$ sale $\theta = \arcsin x$; i de $\sin 2\theta = 2 \sin \theta \cos \theta = 2x \sqrt{1-x^2}$:

$$\int \sqrt{1-x^2} dx = \frac{1}{2} (\arcsin x + x \sqrt{1-x^2})$$

(Edwards, 38)

2. $\int \sqrt{1+x^2} dx$. Hacemos $x = \tan \theta \therefore dx = \sec^2 \theta d\theta$
 $y = \int \sec^3 \theta d\theta$. Esta integral se calculó en el ejercicio 44 del número 58. Reemplazamos ahí $\tan \theta$ por x :

$$\int \sqrt{1+x^2} dx = \frac{1}{2} L(x + \sqrt{x^2+1}) + \frac{1}{2} x \sqrt{x^2+1}$$

(Todhunter, 11)

3. $\int \sqrt{x^2-1} dx$. Hacemos $x = \sec \theta$

$$\therefore dx = \sec \theta \operatorname{tg} \theta d\theta, \operatorname{tg} \theta = \sqrt{x^2-1}$$

$$y = \int \sec \theta \operatorname{tg}^2 \theta d\theta = \int \sec \theta d\theta + \int \sec^3 \theta d\theta.$$

Estas dos integrales son conocidas.

$$\int \sqrt{x^2-1} dx = \frac{1}{2} x \sqrt{x^2-1} - \frac{1}{2} L(x + \sqrt{x^2-1})$$

(Id., 11)

En la integración por partes encontraremos métodos mas expeditos.

4. $\int \frac{dx}{\sqrt{1-x^2}}$. Sea $x = \operatorname{sen} y \therefore dx = \cos y dy$,

$$\cos y = \sqrt{1-x^2}.$$

$$y = \int \frac{\cos y dy}{\cos y} = \int dy = y = \operatorname{arc} \operatorname{sen} x.$$

5. $\int \frac{dx}{x \sqrt{x^2-a^2}}$. Sea $\frac{a}{x} = \cos y$

$$\begin{aligned} \therefore \frac{\sqrt{x^2 - a^2}}{x} &= \operatorname{sen} y, \int \frac{dx}{x \sqrt{x^2 - a^2}} = \int \frac{x^2}{a} \cdot \frac{\operatorname{sen} y}{\operatorname{sen} y} \cdot \frac{dy}{x^2} \\ &= \frac{1}{a} \int dy = \frac{y}{a} = \frac{1}{a} \arccos \frac{a}{x} = \frac{1}{a} \arcsen \frac{x}{a} \\ &\quad (\text{H. Cox, 35}) \end{aligned}$$

$$6. \int \frac{dx}{\sqrt{(a-x)(x-\beta)}} \quad \text{Sea } x = a \operatorname{sen}^2 \phi + \beta \cos^2 \phi$$

$$\therefore a-x = (a-\beta) \cos^2 \phi, \quad x-\beta = (a-\beta) \operatorname{sen}^2 \phi, \quad (1)$$

$$(a-\beta)(x-\beta) = (a-\beta)^2 \operatorname{sen}^2 \phi \cos^2 \phi$$

$$dx = 2(a-\beta) \operatorname{sen} \phi \cos \phi d\phi$$

$$y = \int \frac{2(a-\beta) \operatorname{sen} \phi \cos \phi d\phi}{(a-\beta) \operatorname{sen} \phi \cos \phi} = \int 2 d\phi = 2\phi$$

Dividiendo (1):

$$\frac{\operatorname{sen}^2 \phi}{\cos^2 \phi} = \frac{x-\beta}{a-x} \quad \therefore \operatorname{tg} \phi = \sqrt{\frac{x-\beta}{a-x}}$$

$$\therefore \phi = \arctg \sqrt{\frac{x-\beta}{a-x}}$$

(Roberts, 17)

$$7. \int \frac{x^2 dx}{\sqrt{1-x^2}} \quad \text{Sea } x = \cos z \quad \therefore dx = -\operatorname{sen} z dz$$

$$y = \int \frac{\cos^2 z - \sin z}{\sin z} dz = - \int \cos^2 z dz = - \frac{1}{2} z - \frac{1}{4} \sin^2 z.$$

$$\sin 2z = 2 \sin z \cos z = 2 \sqrt{1 - \cos^2 z} \cos z = 2 \sqrt{1 - x^2} x.$$

$$\therefore y = - \frac{1}{4} \arccos x - \frac{1}{2} x \sqrt{1 - x^2}$$

(J. Bertrand, II, 8)

$$8. \int \frac{dx}{(x^2 + 2bx + c)^n}. \text{ Hacemos } x + b = \sqrt{c - b^2} \operatorname{tg} t$$

$$dx = \sqrt{c - b^2} \sec^2 t dt$$

$$x^2 + 2bx + b^2 = (c - b^2) \operatorname{tg}^2 t$$

$$x^2 + 2bx = (c - b^2) \operatorname{tg}^2 t - b^2$$

$$x^2 + 2bx + c = (c - b^2) \operatorname{tg}^2 t + (c - b^2)$$

$$= (c - b^2) (1 + \operatorname{tg}^2 t)$$

$$= (c - b^2) \sec^2 t$$

$$y = \frac{1}{c - b^2} \int \cos^{n-2} t dt.$$

En las reducciones sucesivas se dará la integral.

(Tannery, 521)

64. *Racionalizacion.*—Operacion, cuyo objeto es hacer racional una expresion irracional. Se emplea para ésto el método de sustitucion.

I. *Funciones monomias.*

1. $y = \int \sqrt{x} \, dx$. Hacemos $x = z^2 \therefore dx = 2z \, dz$

$$\therefore y = \int 2z^2 \, dz = \frac{2}{3} z^3 = \frac{2}{3} x^{\frac{3}{2}}$$

II. *Suma de radicales.* La variable auxiliar tiene por esponente el menor múltiplo comun de los índices.

$$2. \int \left(\sqrt{x} + \sqrt[3]{x} \right) dx = \int \left(x^{\frac{1}{2}} + x^{\frac{1}{3}} \right) dx.$$

El menor múltiplo comun de los índices o denominadores 2 i 3, es 6:

Hacemos

$$x = z^6 \therefore dx = 6z^5 \, dz.$$

Sustituyamos i obtenemos la funcion de términos racionales:

$$y = \int (z^3 + z^2) 6z^5 \, dz = 6 \int (z^8 + z^7) \, dz.$$

$$= \frac{2}{3} x^{\frac{3}{2}} + \frac{3}{4} x^{\frac{4}{3}}$$

Este método se emplea solamente en las fracciones polinomias de términos irracionales.

III. *Fracciones de términos irracionales.*

$$3. \quad y = \int \frac{\sqrt{x}}{\sqrt{x}-1} dx. \quad (\text{Francoeur, II, 364})$$

Sea

$$x = z^2 \quad \therefore \quad dx = 2z dz;$$

queda:

$$\begin{aligned} y &= \int \frac{2z^2}{z^2-1} dz = \int \left(1 + \frac{1}{z^2-1} \right) dz \\ &= 2x + L \frac{\sqrt{x-1}}{\sqrt{x+1}} \end{aligned}$$

$$4. \quad y = \int \frac{x^{\frac{3}{2}} - x^{\frac{5}{6}}}{x^{\frac{1}{4}}} dx. \quad \text{Efectuamos la division; o}$$

hacemos $x=z^n$, siendo n el menor múltiplo comun de los índices; aquí $n=12$:

Sea

$$x = z^{12} \quad \therefore \quad dx = 12 z^{11} dz, \quad x^{\frac{3}{2}} = z^{18},$$

$$x^{\frac{5}{6}} = z^{10}, \quad x^{\frac{1}{4}} = z^3;$$

queda:

$$y = 12 \int \frac{z^{18} - z^{10}}{z^3} z^{11} dz = 12 \int (z^{18} - z^{10}) z^8 dz$$

$$= 12 \int (z^{27} - z^{18}) dz = 12 x \frac{9}{4} - x \frac{19}{12}.$$

$$5. \quad \int \frac{1 + \sqrt[3]{x}}{1 + \sqrt{x}} = \int \frac{1 + x^{\frac{1}{2}}}{1 + x^{\frac{2}{3}}}. \quad \text{Hacemos } x = z^6$$

$$y = 6 \int \frac{z^2 + z^5}{z^2 + 1} dz.$$

Efectuamos la division e integramos en seguida:

$$y = \frac{6}{7} x^{\frac{7}{6}} - \frac{6}{5} x^{\frac{5}{6}} + \frac{3}{2} x^{\frac{2}{3}} + 2x^{\frac{1}{2}} - 3x^{\frac{1}{3}} - 6x^{\frac{1}{6}} \\ + 3L\left(x^{\frac{1}{3}} + 1\right) + 6 \operatorname{arc} \operatorname{tg} x^{\frac{1}{6}}.$$

(Serret, II, 24)

$$6. \quad y = \int \frac{x^{\frac{1}{m}} + x^{\frac{2}{n}}}{x^p - x^{\frac{r}{s}}} dz. \quad \text{Hacemos } x = z^{mns}.$$

$$y = \int \frac{z^{ns} + z^{2ms}}{z^{p m n s} - z^{m n r}} dz.$$

En seguida se descompone en dos fracciones, se divide o se trasforma en fracciones parciales.

$$7. \quad \int \frac{1 + x^{\frac{1}{4}}}{1 + x^{\frac{1}{3}}} dx = \frac{1}{11} x^{\frac{11}{12}} + \frac{1}{8} x^{\frac{3}{2}} - \frac{1}{7} x^{\frac{7}{12}} - \frac{1}{4} x^{\frac{1}{5}}$$

$$+ \frac{1}{3} x \frac{1}{4} + \frac{1}{2 \frac{5}{2}} \left[L \frac{x \frac{1}{6} - 2 \frac{1}{2} x \frac{1}{12} + 1}{x \frac{1}{6} + 2 \frac{1}{2} x \frac{1}{12} + 1} + 2 \operatorname{arc} \operatorname{tg} \frac{2 \frac{1}{2} x \frac{1}{12}}{1 - x \frac{1}{6}} \right]$$

(Grégory, 267)

$$8. \quad dy = \frac{1 + \sqrt{x} + \sqrt[3]{x^2}}{1 + \sqrt{x}} dx. \text{ Sea } x = t^6 \therefore dx = 6t^5 dt.$$

Sustitúyese i resulta la fraccion de términos racionales:

$$dy = \frac{1 + t^3 - t^4}{1 + t^2} \cdot 6 t^5 dt.$$

Efectuamos la division:

$$y = 6 \int (-t^7 + t^6 + t^5 - t^4 + t^2 - 1 + \frac{1}{1+t^2}) dt \\ = -\frac{3}{4} t^8 + \frac{6}{7} t + t^6 - \frac{6}{5} t^5 + 2 t^3 - 6t + 6 \operatorname{arc} \operatorname{tg} t.$$

(Sturm, 334)

$$9. \quad dy = \frac{\sqrt[3]{x} - \sqrt[6]{x} + 1}{\sqrt{x} - 1} dx. \text{ Sea } x = z^6$$

$$y = 6 \left[\frac{1}{7} x^{\frac{7}{6}} + \frac{1}{5} x^{\frac{5}{6}} + \frac{1}{4} x^{\frac{2}{3}} + \frac{1}{3} x^{\frac{1}{2}} + \frac{1}{2} x^{\frac{1}{3}} + x^{\frac{1}{6}} + L \left(x^{\frac{1}{6}} - 1 \right) \right]$$

(Brahya, 21)

$$10. \quad dy = \frac{\sqrt{x} - 1}{6 \left(\sqrt[3]{\sqrt{x} + 1} \right)} dx. \quad \text{Sea } x = z^6$$

$$y = \frac{1}{7} x^{\frac{7}{6}} - \frac{1}{5} x^{\frac{5}{6}} - \frac{1}{4} x^{\frac{2}{3}} + \frac{1}{3} x^{\frac{1}{2}} + \frac{1}{2} x^{\frac{1}{3}} - x^{\frac{1}{6}} + \arctg x^{\frac{1}{6}} - L \sqrt{x^{\frac{1}{3}} + 1}.$$

$$11. \quad dy = \frac{x^2 + \sqrt[3]{(ax + b)^2}}{x + \sqrt{ax + b}} dx. \quad \text{Sea } ax + b = z^6$$

$$y = \frac{6}{a^2} \int \frac{z^5 [(z^6 - b)^2 + a^2 b^4]}{z^6 - b + a z^3} dz.$$

Se integra por division.

$$12. \quad dy = \frac{\sqrt[3]{x} + x \sqrt{x} + x^2}{x + \sqrt{x}} dx. \quad \text{Sea } x = z^6$$

$$y = \int \left(z^{11} + z - \frac{z}{z^3 + 1} \right) dz.$$

Cada término es de fácil integracion.

$$13. \quad dy = \frac{a \sqrt[3]{x^2} - b \sqrt[4]{x^5}}{a \sqrt[3]{x^3} + \sqrt[3]{x^2} + e x^2} dx. \quad \text{Sea } x = z^{12}$$

$$y = 12 \int \frac{a - b z^7}{c z^7 + e z^5} dz$$

Se integra por el método de las fracciones racionales.

(Timmermans, 271)

IV. Funciones irracionales.

$$14. \quad dy = \left(\frac{ax+b}{a^1x+b^1} \right)^{\frac{m}{n}} dx. \quad \text{Sea } \frac{ax+b}{a^1x+b^1} = z^n$$

$$\therefore dx = -\frac{(b^1 - a b^1) n z^{n-1}}{(b - b^1 z^n)^2}$$

$$y = n (b a^1 - a b^1) \int \frac{z^{m+n-1}}{(b - b^1 z^n)^2} dz$$

Esta funcion racional es de fácil integracion.

$$15. \quad \int x^m (a + b x^n)^{\frac{p}{q}} \cdot \text{Esta forma lleva el nombre de diferencial binomia i tiene una gran importancia.}$$

Se puede racionalizar cuando sea entero $\frac{m+1}{n}$; en tal caso se hace $a + b x^n = z^q$; i si $\frac{m+1}{n} + \frac{p}{q}$ es entero, se hace $a + b x^n = x^n z^q$.

La siguiente aplicacion es de Grégory, páj. 265.

$$16. \int \frac{x}{(a+bx)^{\frac{1}{2}}} dx = \int x(a+bx)^{-\frac{1}{2}} dx$$

Aquí $m=1$

$n=1$

$\therefore \frac{m+1}{n} = 2$, hacemos $a+bx = z^2$

$$\frac{p}{q} = -\frac{1}{2}$$

$$dx = \frac{2z}{b} dz, x = \frac{b^2 - a}{b},$$

$$y = \frac{2}{b^2} \int (z^2 - a) dz = \frac{2}{b^2} \left(\frac{1}{3} z^3 - az \right)$$

$$= \frac{2}{3b^2} (bx - 2a) \sqrt{a+bx}$$

$$17. \int x^3(a+x)^{\frac{2}{3}}$$

$m=3$

$n=1$

$\therefore \frac{m+1}{n} = 4 \therefore a+x = z^3, dx = 3z^2 dz$

$$\frac{p}{q} = \frac{2}{3}$$

$q=3$

$$y = \int (z^3 - a)^2 z^2 dz = \int (z^{11} - 3az^8 + 3a^2z^5 - a^3z^2) dz$$

$$= (a+x)^{\frac{5}{3}} \left[\frac{1}{16} (a+x)^{\frac{11}{3}} - \frac{a}{13} (a+x)^{\frac{8}{3}} + \frac{3a^2}{10} (a+x)^{\frac{5}{3}} \right]$$

$$-\frac{a^3}{7} (a+x)^{\frac{2}{3}} \Big] .$$

$$18. \int \frac{dx}{x^4(1+x^2)^{\frac{1}{2}}} = \int x^{-4}(1+x^2)^{-\frac{1}{2}} dx$$

$$\begin{array}{l} m = -4 \\ n = 2 \end{array} \left| \begin{array}{l} \frac{m+1}{n} + \frac{p}{q} = -2 \dots 1+x^2 = x^2 z^2 \\ \frac{p}{q} = -\frac{1}{2} \end{array} \right|$$

$$dx = \frac{xz}{1-z^2} dz, \quad x^4 = \frac{1}{(z^2-1)^2}$$

$$y = \int \frac{\frac{xz dz}{1-z^2}}{\frac{xz}{(z^2-1)^2}} = \int (1-z^2) dz = z - \frac{1}{3} z^3$$

$$= \sqrt{1+x^2} \left(\frac{1}{x} - 1 - 3x \right)$$

$$19. \quad w = \int \frac{dx}{(1+nx^2)\sqrt{1-x^2}} \quad \text{Sea } t = \frac{x}{\sqrt{1-x^2}}$$

$$dx = \frac{dt}{(1+t^2)^{\frac{3}{2}}} \quad w = \int \frac{\frac{dt}{(1+t^2)^{\frac{3}{2}}}}{\left(1 + \frac{nt^2}{1+t^2}\right) \left(\frac{1}{1+t^2}\right)^{\frac{1}{2}}}$$

Simplificando, se obtiene:

$$w = \int \frac{dt}{1+(1+n)t^2} = \frac{1}{\sqrt{1+n}} \operatorname{arc} \operatorname{tg} \sqrt{1+n} t$$

$$= \frac{1}{\sqrt{1+n}} \operatorname{arc} \operatorname{tg} \frac{x}{\sqrt{1-x^2}}$$

(Serret, 49)

20. $\int \sqrt{a+bx+x^2}$. Sea $\sqrt{a+bx+x^2} = z-x$

Elevando al cuadrado i diferenciando resulta:

$$y=2 \int \frac{(a+bz+z^2)^2}{(b+xz)^3} dz.$$

65. *Conversion de las funciones.*—Bajo este nombre designamos la trasformacion que convierte una funcion irracional en racional o una trascendente en algebraica.

En los dos números anteriores se dieron numerosos ejemplos de esta clase de trasformacion.

Agregaremos otros mas, a fin de que el principiante se familiarice con estos cambios de formas de las funciones.

1. $dy = L^n x \frac{dx}{x}$. Sea $Lx = z \therefore \frac{dx}{x} = dz$

La funcion logarítmica se convierte en la racional

$$y = \int z^n dx = \frac{1}{n+1} L^{n+1} x.$$

2. $\int \operatorname{sen} x \, dx$. Sea $\operatorname{tg} \frac{1}{2} = z = \sqrt{\frac{1-\cos x}{1+\cos x}}$.

Los dos primeros miembros dan:

$$a = 2 \operatorname{arc} \operatorname{tg} z \quad \therefore \quad dx = \frac{2 \, dz}{1+z^2};$$

i los dos últimos,

$$\cos x = \frac{1-z^2}{1+z^2}, \quad \operatorname{sen} x = \frac{2z}{1+z^2}$$

Sustituyendo:

$$\begin{aligned} \int \operatorname{sen} x \, dx &= \int \frac{2z}{1+z^2} \cdot \frac{2 \, dz}{1+z^2} = 2 \int (1+z^2)^{-2} \, d(1+z^2) \\ &= -\frac{2}{1+z^2} = -1 - \cos x. \end{aligned}$$

Difiere esta integral en -1 de la integral conocida. La auxiliar anterior se emplea en ejemplos mas complicados, como los que siguen:

3. $\int \frac{d\theta}{a+b \cos \theta}$. Sea $\cos \theta = \frac{1-z^2}{1+z^2} \therefore d\theta = \frac{2 \, dz}{1+z^2}$.

$$= \int \frac{\frac{2 \, dz}{1+z^2}}{a + \frac{b(1-z^2)}{1+z^2}} = \frac{dz}{a+b} \int \frac{dz}{1 + \frac{a-b}{a+b} z^2}$$

si $a > b$, tendremos:

$$y = \frac{2}{\sqrt{(a+b)(a-b)}} \arctg \sqrt{\frac{a-b}{a+b}} z$$

$$= \frac{2}{\sqrt{a^2-b^2}} \arccos \frac{a \cos \theta + b}{a + b \cos \theta}$$

si $a < b$:

$$y = \frac{1}{\sqrt{b^2-a^2}} L \frac{\operatorname{tg} \frac{1}{2} \theta + \sqrt{\frac{b+a}{b-a}}}{\operatorname{tg} \frac{1}{2} \theta - \sqrt{\frac{b+a}{b-a}}}$$

si $a=b$:

$$y = \frac{1}{a} \operatorname{tg} \frac{1}{2} \theta$$

(Sturm, 364)

4. $\int \frac{dx}{a \sin x + b \cos x + c}$. El método anterior nos da:

$$y = \int \frac{\frac{2 dz}{1+z^2}}{a \cdot \frac{2z}{1+z^2} + \frac{b(1-z^2)}{1+z^2} + \frac{c(1+z^2)}{1+z^2}}$$

$$= \int \frac{2 dz}{(c-b)z^2 + 2az + b+c}$$

Esta forma es conocida.

$$5. \quad y = \int \sin^m x \cos^n x \, dx. \quad \text{Sea } z = \sin^2 x$$

$$\therefore \cos^2 x = 1 - z, \quad dz = 2 \sin x \cos x \, dx$$

$$y = \pm \frac{1}{2} \int z^{\frac{m-1}{2}} (1-z)^{\frac{n-1}{2}} \, dz$$

$$6. \quad y = \int \sin^4 x \cos^3 x \, dx. \quad \text{Segun lo anterior:}$$

$$y = \int z^2 (1-z^2) = \frac{1}{5} \sin^5 x - \frac{1}{7} \sin^7 x.$$

(Francoeur, 378)

$$7. \quad y = \int f(\sin x, \cos x) \, dx. \quad \text{Sea } x = z$$

$$\therefore \cos x = \sqrt{1-z^2}, \quad dx = \frac{dz}{\sqrt{1-z^2}}$$

Se obtiene la funcion algebraica:

$$y = \int f(z, \sqrt{1-z^2}) \frac{dz}{\sqrt{1-z^2}}$$

(Moigno, II, 33)

$$7. \quad y = \int \arcsin x \, dx. \quad \text{Convertimos esta funcion circular en trigonométrica, haciendo } \arcsin x = z \therefore dz = \frac{dx}{\sqrt{1-x^2}}.$$

$$y = \int z \cos z \, dz.$$

66. *Empleo de las imaginarias.*—1. Para construir una función circular en logaritmica, hacemos $a = 1$, $x = iz$, $i^2 = -1$, $dx = i dz$, en la integral

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{a} L \sqrt{\frac{a+x}{a-x}};$$

se obtiene:

$$\int \frac{i dz}{1+z^2} = \frac{1}{2} L \frac{1+iz}{1-iz}$$

o bien

$$i \operatorname{arc} \operatorname{tg} z = \frac{1}{2} L \frac{1+iz}{1-iz}$$

(Roberts, 14)

2. $\int e^{ax} \cos bx \, dx$. Empleemos las ecuaciones de Euler:

$$\operatorname{sen} x = \frac{e^{ix} - e^{-ix}}{2i}, \quad \cos x = \frac{e^{ix} + e^{-ix}}{2};$$

$$y = \int e^{ax} \cdot \frac{e^{bix} + e^{-bix}}{2} dx = \int \frac{e^{(bi+a)x} + e^{(a-bi)x}}{2} dx$$

$$= \frac{1}{2} \left[\frac{e^{(a+bi)x}}{a+bi} + \frac{e^{(a-bi)x}}{a-bi} \right]$$

$$= \frac{a \cos bx + b \operatorname{sen} bx}{a^2 + b^2} e^{ax}$$

$$3. \int e^{ax} \sin bx \, dx = \frac{a \sin bx - b \cos bx}{a^2 + b^2} e^{ax}$$

(Tannery, II, 510)

$$4. \, dy = \frac{-dx}{\sqrt{1-x^2}} = \frac{-dx}{i\sqrt{x^2-1}}$$

$$\therefore yi = L(x + \sqrt{x^2-1})$$

Sea

$$x = \cos y \quad \therefore x^2 - 1 = \cos^2 y - 1 = -\sin^2 y;$$

queda

$$\pm yi = L(\cos y \pm i \sin y).$$

Para demostrar que este resultado es cierto, acudimos a las dos ecuaciones de Euler:

$$\cos y = \frac{e^{iy} + e^{-iy}}{2}$$

$$i \sin y = \frac{e^{iy} - e^{-iy}}{2}$$

Si sumamos miembro a miembro, se obtiene:

$$\cos y + i \sin y = e^{iy};$$

Aplicamos logaritmos:

$$iy = L(\cos y + i \sin y).$$

CAPÍTULO VI

INTEGRACION POR PARTES

67. *Idea jeneral del método.*—En el número 34 dimos a conocer la fórmula empleada:

$$\int u \, dv = u \, v - \int v \, du,$$

que comprende la sustitucion de u i v como funciones de x , i la descomposicion de un producto en una diferencia.

Este método se emplea en las funciones irracionales i trascendentes.

Para el objeto de comprender todos sus recursos e importancia, hemos considerado diversos casos.

El mas sencillo es cuando la *nueva integral* $\int v \, du$ es de integracion inmediata.

68. *La nueva integral es de integracion inmediata.*

1. $y = \int L \, x \, dx.$

Sea

$$u = L \, x \quad \therefore \quad du = \frac{dx}{x},$$

$$dv = dx \quad \therefore \quad v = x.$$

Aquí se hizo la diferenciacion de $u = L \, x$ i la integracion de $dv = dx$.

Sustituyamos en la fórmula:

$$\begin{cases} \int u \, dv = uv - \int v \, du \\ \int Lx \, dx = x Lx - \int dx \end{cases}$$

La nueva integral $\int dx$ es de integracion inmediata:

$$\int Lx \, dx = x Lx - x.$$

2. $y = \int \arcsen x \, dx.$

Sea

$$u = \arcsen x \quad \therefore \quad du = \frac{dx}{\sqrt{1-x^2}}$$

$$dv = dx \quad \therefore \quad v = x$$

$$y = \int \arcsen x \, dx = x \arcsen x - \int \frac{x \, dx}{\sqrt{1-x^2}}$$

Aquí

$$\int \frac{x \, dx}{\sqrt{1-x^2}} = -\frac{1}{2} \int (1-x^2)^{-\frac{1}{2}} d(1-x^2) = -\sqrt{1-x^2}.$$

$$\therefore \int \arcsen x \, dx = x \arcsen x + \sqrt{1-x^2}$$

3. $\int \arccos x \, dx.$

$$\left| \begin{array}{l} u = \arccos x \quad \therefore \quad du = \frac{-dx}{\sqrt{1-x^2}} \\ dv = dx \quad \therefore \quad v = x \end{array} \right.$$

$$y = x \operatorname{arc} \cos x + \int \frac{x dx}{\sqrt{1-x^2}} = x \operatorname{arc} \cos x - \sqrt{1-x^2}$$

$$4. \int \operatorname{arc} \operatorname{tg} x dx. \quad \left| \begin{array}{l} u = \operatorname{arc} \operatorname{tg} x \quad \therefore du = \frac{dx}{1+x^2} \\ dx = dx \quad \therefore v = x \end{array} \right.$$

$$y = x \operatorname{arc} \operatorname{tg} x - \int \frac{x dx}{1+x^2} = x \operatorname{arc} \operatorname{tg} x - \frac{1}{2} L(1+x^2)$$

(Humbert, I, 429)

$$5. \int \operatorname{arc} \cot x dx = x \operatorname{arc} \cot x + L(1+x^2) \frac{1}{2}$$

$$6. \int \operatorname{arc} \sec x dx = x \operatorname{arc} \sec x - L(x + \sqrt{x^2-1})$$

$$7. \int \operatorname{arc} \operatorname{cosec} x dx = x \operatorname{arc} \operatorname{cosec} x + L(x + \sqrt{x^2-1})$$

$$8. \int \operatorname{arc} \operatorname{vers} x dx = x \operatorname{arc} \operatorname{vers} x - \int \frac{x dx}{\sqrt{2x-x^2}}$$

$$\int \frac{-x dx}{\sqrt{2x-x^2}} = \int \frac{-x+1-1}{\sqrt{2x-x^2}} dx = \sqrt{2x-x^2} - \operatorname{arc} \operatorname{vers} x$$

$$\therefore \int \operatorname{arc} \operatorname{vers} x dx = (x-1) \operatorname{arc} \operatorname{vers} x + \sqrt{2x-x^2}$$

(Brahya, 33)

$$9. \int (1+\operatorname{tg}^2 x) dx = \int dx + \int \operatorname{tg}^2 x dx.$$

$$\operatorname{tg}^2 x = \frac{\operatorname{sen}^2 x}{\cos^2 x} = - \frac{\operatorname{sen} x d \cos x}{\cos^2 x} :$$

Sea

$$u = \operatorname{sen} x \quad \therefore \quad du = \cos x \, dx,$$

$$dv = \frac{d \cos x}{\cos^2 x} \quad \therefore \quad v = \frac{1}{\cos x};$$

sustituimos:

$$\int_x \operatorname{tg}^2 x = \frac{\operatorname{sen} x}{\cos x} - \int dx = \operatorname{tg} x - x;$$

$$\therefore \int (1 + \operatorname{tg}^2 x) = x + \operatorname{tg} x - x = \operatorname{tg} x.$$

$$10. \int_x (1 + \cot^2 x) = \int dx + \int \cot^2 x \, dx, \text{ o bien,}$$

$$= - \int \left[1 + \operatorname{tg}^2 \left(\frac{\pi}{2} - x \right) \right] d \left(\frac{\pi}{2} - x \right) = - \operatorname{tg} \left(\frac{\pi}{2} - x \right)$$

$$= - \cot x.$$

$$11. \int x e^x \, dx. \quad \text{Sea } u = x \quad \therefore \quad du = dx$$

$$dv = e^x \, dx \quad \therefore \quad v = e^x$$

$$y = x e^x - \int e^x \, du = x e^x - e^x$$

$$12. \int x L x \, dx. \quad \left| \begin{array}{ll} u = L x & \therefore \quad du = \frac{dx}{x} \\ dv = x \, dx & \therefore \quad v = \frac{1}{2} x^2 \end{array} \right.$$

$$y = \frac{1}{2} x^2 L x - \int \frac{1}{2} x^2 \cdot \frac{d x}{x} = \frac{1}{2} x^2 L x - \frac{1}{4} x^2$$

(Appell, 51)

$$13. \quad \int x^3 L x d x. \quad \left| \begin{array}{ll} u = L x & \therefore d u = \frac{d x}{x} \\ d v = x^3 d x & v = \frac{1}{4} x^4 \end{array} \right.$$

$$y = \frac{1}{4} x^4 L x - \int \frac{1}{4} x^4 \cdot \frac{d x}{x} = \frac{1}{4} x^4 L x - \frac{1}{16} x^4$$

(Bertrand, 9)

$$14. \quad \int L x \cdot x^n d x. \quad \left| \begin{array}{ll} u = L x & \therefore d u = \frac{d x}{x} \\ d v = x^n d x & \therefore v = \frac{x^{n+1}}{n+1} \end{array} \right.$$

$$y = \frac{x^{n+1}}{n+1} L x - \frac{1}{n+1} \int x^n d x$$

$$= \frac{x^{n+1}}{n+1} \left(L x - \frac{1}{n+1} \right)$$

(Briot, II, 504)

$$15. \quad \int x \operatorname{sen} x d x. \quad \left| \begin{array}{ll} u = x & \therefore d u = d x \\ d v = \operatorname{sen} x d x & \therefore v = -\cos x \end{array} \right.$$

$$y = -x \cos x + \int d \operatorname{sen} x = -x \cos x + \operatorname{sen} x$$

(Moigno, II, 15)

$$16. \int x \cos x \, dx = x \operatorname{sen} x - \int \operatorname{sen} x \, dx$$

$$= x \operatorname{sen} x + \cos x$$

(Duhamel, I, 429)

$$17. \int x \cos ax \, dx = x \int \frac{\operatorname{sen} ax}{a} - \frac{1}{a} \int \operatorname{sen} ax \, dx$$

$$= \frac{x \operatorname{sen} ax}{a} + \frac{\cos ax}{a^2}$$

$$18. \int \cos x \operatorname{L} \operatorname{sen} x \, dx. \quad \left| \begin{array}{l} u = \operatorname{L} \operatorname{sen} x \quad \therefore \quad du = \cot x \\ dv = \cos x \, dx \quad \therefore \quad v = -\operatorname{sen} x \end{array} \right.$$

$$y = -\operatorname{sen} x \operatorname{L} \operatorname{sen} x - \operatorname{sen} x.$$

$$19. \int \operatorname{sen} x \operatorname{L} \cos x \, dx = \cos x \operatorname{L} \cos x + \cos x.$$

$$20. \int x \cdot \operatorname{arc} \operatorname{sen} x \, dx. \quad \left| \begin{array}{l} u = \operatorname{arc} \operatorname{sen} x \quad \therefore \quad du = \frac{dx}{\sqrt{1-x^2}} \\ v \, dv = x \, dx \quad \therefore \quad v = \frac{1}{2} x^2 \end{array} \right.$$

$$y = \frac{1}{2} x^2 \operatorname{arc} \operatorname{sen} x - \frac{1}{4} \int \frac{x^2 \, dx}{\sqrt{1-x^2}}$$

$$= \frac{1}{2} x^2 \operatorname{arc} \operatorname{sen} x + \frac{1}{4} \operatorname{arc} \cos x - \frac{1}{4} x \sqrt{1-z^2}$$

(Bertrand. 10)

$$21. \int \frac{x \operatorname{arc} \operatorname{sen} x}{\sqrt{1-x^2}} dx \quad \left| \begin{array}{l} u = \operatorname{arc} \operatorname{sen} x \quad \therefore du = \frac{dx}{\sqrt{1-x^2}} \\ dv = \frac{x dx}{\sqrt{1-x^2}} \quad \therefore v = -\sqrt{1-x^2} \end{array} \right.$$

$$y = -\sqrt{1-x^2} \operatorname{arc} \operatorname{sen} x + x.$$

$$22. \int f(x) \operatorname{arc} \operatorname{sen} x dx = \operatorname{arc} \operatorname{sen} x \int f(x) dx$$

$$- \int \left(\frac{dx}{\sqrt{1-x^2}} \int f(x) dx \right)$$

La integracion por partes transforma una funcion trascendente en algebrica. (Francoeur, 378)

$$23. \int \frac{x(3-x^2) \operatorname{arc} \operatorname{tg} x}{(1-x^2)^{\frac{3}{2}}} dx.$$

$$u = \operatorname{arc} \operatorname{tg} x \quad \therefore du = \frac{dx}{1+x^2}$$

$$dv = \frac{x(3-x^2)}{(1+x^2)^{\frac{3}{2}}} dx = \frac{x(2+1-x^2)}{(1-x^2)^{\frac{3}{2}}} dx$$

$$= - (1-x^2)^{-\frac{3}{2}} d(1-x^2) - \frac{1}{2} (1-x)^{-\frac{1}{2}} d(1-x^2)$$

$$= \frac{1+x^2}{\sqrt{1-x^2}} dx$$

$$\begin{aligned} \therefore y &= \frac{1+x^2}{\sqrt{1-x^2}} \operatorname{arc} \operatorname{tg} x - \int \frac{1+x^2}{\sqrt{1-x^2}} \cdot \frac{dx}{1+x^2} \\ &= \frac{1+x^2}{\sqrt{1-x^2}} \operatorname{arc} \operatorname{tg} x - \operatorname{arc} \operatorname{sen} x. \end{aligned}$$

$$24. \int \frac{a-2x}{\sqrt{ax-x^2}} \operatorname{arc} \operatorname{sen} \sqrt{\frac{a-x}{a+x}} \cdot dx$$

Sea

$$u = \operatorname{arc} \operatorname{sen} \sqrt{\frac{a-x}{a+x}}$$

Para diferenciar, hacemos

$$z = \sqrt{\frac{a-x}{a+x}}$$

$$\therefore du = d \operatorname{arc} \operatorname{sen} y = \frac{dz}{\sqrt{1-z^2}}$$

$$dz = - \frac{a}{(a-x)^{\frac{1}{2}} (a+x)^{\frac{3}{2}}}, \quad 1-z^2 = \frac{2x}{a+x}$$

$$\therefore du = - \frac{a}{\sqrt{3} (ax-x^2)^{\frac{1}{2}} (a+x)}$$

sea

$$dv = \frac{a-2x}{\sqrt{ax-x^2}} dx \quad \therefore v = \sqrt{ax-x^2}$$

$$y = \sqrt{a-x-x^2} \operatorname{arc} \operatorname{sen} \frac{\sqrt{a-x}}{\sqrt{a+x}} + \sqrt{\frac{a}{2}} \int \frac{a}{a+x}$$

La nueva integracion es de inmediata integracion.

$$25. \int \frac{x}{\sqrt{1-x^2}} \operatorname{arc} \operatorname{sec} \sqrt{\frac{1+x^2}{1-x^2}}$$

Sea

$$u = \operatorname{arc} \operatorname{sec} \sqrt{\frac{1+x^2}{1-x^2}}, z = \sqrt{\frac{1+x^2}{1-x^2}}$$

$$\therefore du = d \operatorname{arc} \operatorname{sec} z = \frac{dz}{z \sqrt{z^2 - 1}}$$

$$Lz = \frac{1}{2} [L(1+x^2) - L(1-x^2)]$$

$$\frac{dz}{z} = \frac{1}{2} \left[\frac{2x}{1+x^2} + \frac{2x}{1-x^2} \right] dx$$

$$dz = \frac{x \cdot dx}{\sqrt{1+x^2} \cdot \sqrt{(1-x^2)^3}}$$

$$du = \frac{\frac{x}{\sqrt{1+x^2} \sqrt{(1-x^2)^3}}}{\frac{\sqrt{1+x^2}}{\sqrt{1-x^2}} \cdot \sqrt{\frac{1+x^2}{1-x^2}} - 1}$$

$$= \frac{x(1-x^2)}{x(1+x^2)(1-x^2)^{\frac{3}{2}}} = \frac{1}{(1+x^2)(1+x^2)^{\frac{1}{2}}}$$

Sea

$$d v = \frac{x d x}{\sqrt{1-x^2}} \quad \therefore v = -\sqrt{1-x^2}$$

$$y = -\sqrt{1-x^2} \operatorname{arc} \sec \sqrt{\frac{1+x^2}{1-x^2}} + \operatorname{arc} \operatorname{tg} x.$$

(Brahya, 34)

69. *Trasposicion de la nueva integral.*—Al aplicar la fórmula de la integracion por partes,

$$\int u d v = u v - \int v d u,$$

puede suceder que la nueva integral $\int v d u$ sea igual a la integral primitiva $\int u d v$, o múltiplo de ella; en tal caso tendremos:

$$\int u d v = u v - k \int u d v.$$

Trasponemos la nueva integral:

$$(1+k) \int u d v = u v$$

$$\therefore \int u d v = \frac{1}{1+k} u v.$$

De esta manera queda eliminada la nueva integral.

$$1. \int x^n d x. \quad \text{Sea} \quad \left| \begin{array}{ll} u = x^n & \therefore d u = n x^{n-1} d x \\ d v = d x & v = x \end{array} \right.$$

$$\therefore \int x^n dx = x^n \cdot x - \int n x \cdot x^{n-1} dx$$

$$= x^{n+1} - n \int x^n dx.$$

Trasponiendo la nueva integral, se obtiene:

$$n \int x^n dx + \int x^n dx = x^{n+1}$$

$$\text{o} \quad (n+1) \int x^n dx = x^{n+1}$$

$$\therefore \int x^n dx = \frac{x^{n+1}}{n+1}$$

(Montférier, II, 172)

2. $\int \sqrt{1+x^2} dx$. Descompongamos el radical:

$$\int \sqrt{1+x^2} dx = \int \frac{dx}{\sqrt{1+x^2}} + \int \frac{x^2}{\sqrt{1+x^2}} dx.$$

La primera integral es conocida:

$$\int \frac{dx}{\sqrt{1+x^2}} = L(x + \sqrt{1+x^2})$$

La segunda se integra por partes:

Sea

$$u = x$$

$$du = dx$$

i sea

$$dx = \frac{x}{\sqrt{1+x^2}} dx \quad \dots v = \frac{1}{2} \int (1+x^2)^{\frac{1}{2}} d(1+x^2) \\ = \sqrt{1+x^2}$$

$$\therefore \int x \cdot \frac{x}{\sqrt{1+x^2}} dx = x \sqrt{1+x^2} - \int \sqrt{1+x^2} dx.$$

Sustituyendo:

$$\int \sqrt{1+x^2} dx = L(x + \sqrt{1+x^2}) + x \sqrt{1+x^2} - \int \sqrt{1+x^2} dx$$

Trasponiendo la nueva integral i reduciendo, se obtiene:

$$\int (1+x^2) dx = \frac{1}{2} [L(x + \sqrt{1+x^2}) + x \sqrt{1+x^2}]$$

Serret, 215)

$$3. \int \sqrt{x^2 + a} dx. \text{ Sea } u = \sqrt{x^2 + a} \quad \dots du = \frac{x dx}{\sqrt{x^2 + a}}$$

$$dv = dx \quad \dots v = x.$$

$$\int \sqrt{x^2 + a} dx = x \sqrt{x^2 + a} - \int \frac{x^2}{\sqrt{x^2 + a}} dx$$

Pero

$$\frac{x^2}{\sqrt{x^2 + a}} = \frac{x^2 + a - a}{\sqrt{x^2 + a}} = \frac{x^2 + a}{\sqrt{x^2 + a}} - \frac{a}{\sqrt{x^2 + a}}$$

$$= \sqrt{x^2 + a} - \frac{a}{\sqrt{x^2 + a}} :$$

$$\int \sqrt{x^2 + a} \, dx = x \sqrt{x^2 + a} - \int \frac{a}{\sqrt{x^2 + a}} + \int \frac{a}{\sqrt{x^2 + a}}$$

Trasponiendo i reduciendo:

$$\begin{aligned} \int \sqrt{x^2 + a} \, dx &= \frac{1}{2} \left(x \sqrt{x^2 + a} + \int \frac{a}{\sqrt{x^2 + a}} \, dx \right) \\ &= \frac{1}{2} \left[L(x + \sqrt{a + x^2}) + x \sqrt{x^2 + a} \right] \end{aligned}$$

(Tannery, 523)

$$4. \quad \int \sqrt{x^2 - 1} \, dx = \int \frac{x^2}{\sqrt{x^2 - 1}} \, dx - \int \frac{1}{\sqrt{x^2 - 1}} \, dx \quad (1)$$

La primera integral se integra por partes.

Sea

$$u = x, \, dv = \frac{x \, dx}{\sqrt{x^2 - 1}}$$

$$\therefore du = dx, \, v = \sqrt{x^2 - 1}$$

Sustituyendo:

$$\int \frac{x^2}{\sqrt{x^2 - 1}} \, dx = x \sqrt{x^2 - 1} - \int \sqrt{x^2 - 1} \, dx \quad (2)$$

La segunda integral es conocida:

$$\int \frac{1}{\sqrt{x^2-1}} = L(x + \sqrt{x^2-1}) \quad (3)$$

Sustituimos (2) i (3) en (1):

$$\int \sqrt{x^2-1} \, dx = x \sqrt{x^2-1} - \int \frac{1}{\sqrt{x^2-1}} \, dx - L(x + \sqrt{x^2-1});$$

trasponemos la nueva integral i reducimos:

$$\int \sqrt{x^2-1} \, dx = \frac{1}{2} x \sqrt{x^2-1} - \frac{1}{2} L(x + \sqrt{x^2-1})$$

$$5. \quad \int \sqrt{1-x^2} \, dx = \int \frac{dx}{\sqrt{1-x^2}} - \int \frac{x^2}{\sqrt{1-x^2}} \, dx$$

La primera integral es conocida:

$$\frac{dx}{\sqrt{1-x^2}} = \text{arc sen } x;$$

i la segunda se integra por partes:

Sea

$$u = x, \, dv = \frac{x \, dx}{\sqrt{1-x^2}}$$

$$\therefore du = dx, \, v = -\sqrt{1-x^2}$$

$$\therefore \int x \frac{x dx}{\sqrt{1-x^2}} = -x \sqrt{1-x^2} + \sqrt{1-x^2} dx$$

Despues de reemplazar, trasponer i reducir, se llega a

$$\int \sqrt{1-x^2} dx = \frac{1}{2} (\text{arc sen } x + x \sqrt{1-x^2})$$

$$\begin{aligned} 6. \quad \int \sqrt{a^2-x^2} dx &= a^2 \int \sqrt{1-\left(\frac{x}{a}\right)^2} d \frac{x}{a} \\ &= \frac{a^2}{2} \left(\text{arc sen } \frac{x}{a} + \frac{x}{a^2} \sqrt{a^2-x^2} \right) \end{aligned}$$

$$\begin{aligned} 7. \quad \int \sqrt{a^2-x^2} dx &= a^2 \int \sqrt{1-\left(\frac{x}{a}\right)^2} \cdot a^2 d \frac{x}{a} \\ &= \frac{a^3}{2} \text{arc sen } \frac{x}{a} + \frac{1}{2} x \sqrt{a^2-x^2} \end{aligned}$$

(Cox, 42)

$$8. \quad \int \sqrt{A x^2 + 2 B x + C} dx. \quad (\text{Tannery, 523})$$

Sea

$$u = \sqrt{A x^2 + 2 B x + C} \quad \therefore du = \frac{A x + B}{\sqrt{A x^2 + 2 B x + C}} dx$$

i sea

$$v = x + \frac{B}{A} \quad \therefore dv = dx.$$

Reemplazando i designando por T el subradical:

$$\int \sqrt{T} dx = \frac{1}{A} (A x + B) \sqrt{T} - \frac{1}{A} \int \frac{(x + B)^2}{\sqrt{T}} dx$$

Pero

$$(A x + B)^2 = A (T) + B^2 - A C$$

$$\frac{(A x + B)^2}{\sqrt{T}} = \frac{A T}{\sqrt{T}} + \frac{B^2 - A C}{\sqrt{T}} = A \sqrt{T} + \frac{B^2 - A C}{\sqrt{T}}$$

$$\int \sqrt{T} dx = \frac{1}{A} (A x + B) \sqrt{T} - \frac{1}{A} \int A \sqrt{T}$$

$$- \frac{1}{A} \int \frac{B^2 - A C}{\sqrt{T}} dx.$$

Trasponemos la nueva integral i reducimos:

$$\int \sqrt{T} dx = \frac{1}{2A} (A x + B) \sqrt{T} - \frac{1}{2A} \int \frac{B^2 - A C}{\sqrt{T}} dx.$$

La nueva integral es conocida.

$$9. \int \frac{Lx}{x} dx. \quad \text{Sea} \quad \left| \begin{array}{l} u = Lx \quad \therefore du = \frac{dx}{x} \\ dv = \frac{dx}{x} \quad \therefore v = Lx \end{array} \right.$$

$$y = L^2 x - \int Lx \frac{dx}{x} = \frac{1}{2} L x.$$

$$10. \quad \int \sec^3 x \, dx. \quad \text{Sea} \quad \left| \begin{array}{l} u = \sec x \quad \therefore \quad du = \sec x \operatorname{tg} x \, dx \\ dv = \sec^2 x \, dx \quad \therefore \quad v = \operatorname{tg} x. \end{array} \right.$$

$$\int \sec^3 x \, dx = \sec x \operatorname{tg} x - \int \sec x \operatorname{tg}^2 x \, dx$$

$$= \sec x \operatorname{tg} x - \int \sec^3 x \, dx + \int \sec x \, dx$$

Trasponemos

$$\int \sec^3 x \, dx$$

i reducimos:

$$\int \sec^3 x \, dx = \frac{1}{2} \left(\sec x \operatorname{tg} x + L \sqrt{\frac{1 + \operatorname{sen} x}{1 - \operatorname{sen} x}} \right)$$

$$11. \quad \int x^m L x \, dx. \quad \left| \begin{array}{l} u = L x \quad \therefore \quad du = \frac{dx}{x} \\ dv = x^m \, dx \quad \therefore \quad v = \frac{x^{m+1}}{m+1} \end{array} \right.$$

$$\int x^m L x \, dx = \frac{x^{m+1}}{m+1} L x - \int \frac{x^{m+1}}{(m+1)x} \, dx$$

$$= \frac{x^{m+1}}{m+1} \left(L x - \frac{1}{2} \right)$$

(Davies, Peck, 177)

70. *Eliminacion por reduccion.*—Si despues de aplicar la fórmula

$$\int u \, dv = u v - \int v \, du,$$

encontramos por medio de trasformaciones o descomposiciones que

$$\int u \, dv = u_1 v_1 + \int v \, du,$$

la adición de las integrales anteriores nos dará:

$$2 \int u \, dv = u v + u_1 v_1$$

$$\therefore \int u v = \frac{1}{4} (u v + u_1 v_1)$$

De este modo se elimina la nueva integral, por reduccion.

$$1. \int \sin^2 x \, dx. \quad \text{Sea} \quad \left| \begin{array}{ll} u = \sin x & \therefore du = \cos x \, dx \\ dv = \sin x \, dx & \therefore v = -\cos x \end{array} \right.$$

$$\therefore \int \sin^2 x \, dx = -\sin x \cos x + \int \cos^2 x \, dx;$$

ademas,

$$\underline{\int \sin^2 x \, dx = \int dx - \int \cos^2 x \, dx}$$

Sumando, desaparece la nueva integral:

$$2 \int \sin^2 x \, dx = x - \sin x \cos x$$

$$\therefore \int \sin^2 x \, dx = \frac{1}{2} (x - \sin x \cos x)$$

2. $\int \sqrt{a^2 - x^2} \, dx$. Integramos por partes:

Sea

$$\left| \begin{array}{l} u = \sqrt{a^2 - x^2} \quad \therefore \quad du = \frac{-x \, dx}{\sqrt{a^2 - x^2}} \\ dv = dx \quad \therefore \quad v = x \end{array} \right.$$

$$\int \sqrt{a^2 - x^2} \, dx = x \sqrt{a^2 - x^2} + \int \frac{x^2 \, dx}{\sqrt{a^2 - x^2}};$$

pero,

$$\int \sqrt{a^2 - x^2} \, dx = a^2 \int \frac{dx}{\sqrt{a^2 - x^2}} - \int \frac{x^2 \, dx}{\sqrt{a^2 - x^2}}.$$

Sumamos i dividimos por 2:

$$\sqrt{a^2 - x^2} \, dx = x \sqrt{a^2 - x^2} + a^2 \arcsen \frac{x}{a}$$

(Clausen, 1129)

$$3. \quad \int \sqrt{x^2 + a^2} \, dx = x \sqrt{x^2 + a^2} - \int \frac{x^2 \, dx}{\sqrt{x^2 + a^2}},$$

$$\int \sqrt{x^2 + a^2} \, dx = a^2 \int \frac{dx}{\sqrt{x^2 + a^2}} - \int \frac{x^2 \, dx}{\sqrt{x^2 + a^2}}:$$

Sumamos:

$$\int \sqrt{x^2 + a^2} \, dx = \frac{x \sqrt{x^2 + a^2}}{2} + \frac{a^2}{2} \ln [x + \sqrt{x^2 + a^2}]$$

$$4. \int \sqrt{x^2 + a^2} \, dx = - \int \frac{x^2}{\sqrt{x^2 - a^2}} \, dx + x \sqrt{x^2 - a^2},$$

$$\int \sqrt{x^2 - a^2} \, dx = \int \frac{x^2}{\sqrt{x^2 - a^2}} \, dx - a^2 \int \frac{dx}{\sqrt{x^2 - a^2}}.$$

Sumamos:

$$\int \sqrt{x^2 - a^2} \, dx = \frac{1}{2} x \sqrt{x^2 - a^2} - \frac{1}{2} a^2 L[x + \sqrt{x^2 - a^2}]$$

(Todhunter, 11)

$$5. \int \frac{x e^{\arcsin x}}{\sqrt{1-x^2}} \, dx. \quad (\text{Brahya, 33})$$

Sea

$$u = e^{\arcsin x} \quad \therefore du = \frac{e^{\arcsin x} dx}{\sqrt{1-x^2}}$$

$$dv = \frac{x}{\sqrt{1-x^2}} \quad \therefore v = -\sqrt{1-x^2}$$

$$\therefore y = -\sqrt{1-x^2} e^{\arcsin x} + e^{\arcsin x} dx \quad (1)$$

Sea ahora:

$$u = x \quad \therefore du = dx$$

$$dv = \frac{e^{\arcsin x}}{\sqrt{1-x^2}} dx \quad v = e^{\arcsin x}$$

$$\therefore y = x e^{\arcsen x} - \int e^{\arcsen x} x dx \quad (2)$$

Sumando i restando las igualdades anteriores (1) i (2), obtenemos:

$$\int \frac{x e^{\arcsen x}}{\sqrt{1-x^2}} dx = \frac{1}{2} e^{\arcsen x} (x - \sqrt{1-x^2}),$$

$$\int e^{\arcsen x} x dx = \frac{1}{2} e^{\arcsen x} (x + \sqrt{1-x^2}).$$

71. *Integracion sucesiva.*—O Doble integracion por partes.
—Cuando la nueva integral de

$$\int u dv = uv - \int v du,$$

no se puede integrar por ninguno de los dos métodos anteriores, hacemos una segunda integracion por partes de $\int v du$.

$$1. \int x^2 e^x dx \quad \text{Sea} \quad \left| \begin{array}{ll} u = x^2 & \therefore du = 2x dx \\ dv = e^x dx & v = e^x \end{array} \right.$$

$$\int x^2 e^x dx = x^2 e^x - 2 \int x \cdot e^x dx$$

Integremos de nuevo por partes:

$$\text{Sea} \quad \left| \begin{array}{ll} u = x & \therefore du = dx \\ dv = e^x dx & v = e^x \end{array} \right.$$

$$\int x^2 e^x dx = x^2 e^x - 2 (x e^x - \int e^x dx)$$

$$= x^2 e^x - 2 x e^x + 2 e^x$$

$$= e^x (x^2 - 2 x + 2)$$

$$2. \quad \int x^3 e^x dx. \quad \left| \begin{array}{ll} u = x^3 & \therefore du = 3 x^2 dx \\ dv = e^x dx & \therefore v = e^x \end{array} \right.$$

$$y = x^3 e^x - 3 \int x^2 e^x dx.$$

Se reduce al ejemplo anterior:

$$y = x^3 e^x - 3 x^2 e^x + 6 x e^x - 6 e^x$$

(De Comberousse, 691)

$$3. \quad \int x^2 a^x dx. \quad \left| \begin{array}{ll} u = x^2 & \therefore du = 2 x dx \\ dv = a^x dx & \therefore v = \frac{a^x}{L a} \end{array} \right.$$

$$y = \frac{x^2 a^x}{L a} - \frac{2}{L a} \int x a^x dx$$

$$\left| \begin{array}{ll} u = x & \therefore du = dx \\ dv = a^x dx & \therefore v = \frac{a^x}{L a} \end{array} \right.$$

$$\int x^2 a^x dx = \frac{x^2 a^x}{L a} - \frac{2}{L a} \left(\frac{x a^x}{L a} - \frac{a^x}{L^2 a} \right)$$

$$= \frac{a^x}{L a} \left(x - \frac{1}{L a} \right)^2$$

$$4. \quad x L^2 x \, dx \quad \left| \quad \begin{array}{l} u = L^2 x \quad \dots \quad du = 2Lx \, dx \\ dv = x \, dx \quad \dots \quad v = \frac{1}{2} x^2 \end{array} \right.$$

$$y = \frac{1}{2} x^2 L^2 x - \int x L x \, dx.$$

$$= \frac{1}{2} x^2 L^2 x - \frac{1}{2} x^2 L x + \frac{1}{4} x^2$$

$$5. \quad \int e^x \cos x \, dx \quad \left| \quad \begin{array}{l} u = \cos x \quad \dots \quad du = -\sin x \, dx \\ dv = e^x \, dx \quad \dots \quad v = e^x \end{array} \right.$$

$$\int e^x \cos x \, dx = e^x \cos x + \int e^x \sin x \, dx. \quad (1)$$

$$\left| \quad \begin{array}{l} u = \sin x \quad \dots \quad du = \cos x \, dx \\ dv = e^x \, dx \quad \dots \quad v = e^x \end{array} \right.$$

$$\int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx \quad (2)$$

De las igualdades (1) i (2), sacamos:

$$\int e^x \cos x \, dx = \frac{1}{2} e^x (\sin x + \cos x)$$

$$\int e^x \sin x \, dx = \frac{1}{2} e^x (\sin x - \cos x)$$

$$6. \quad \int e^{ax} \cos bx \, dx \quad \left| \quad \begin{array}{l} u = \cos bx \quad \dots \quad du = -\sin bx \cdot b \, dx \\ dv = e^{ax} \, dx \quad \dots \quad v = \frac{e^{ax}}{a} \end{array} \right.$$

$$\int e^{ax} \cos bx \, dx = \frac{1}{a} \cos bx \cdot e^{ax} + \frac{b}{a} \int e^{ax} \sin bx \, dx.$$

$$\left| \begin{array}{ll} u = \sin bx & du = b \cos bx \, dx \\ dv = e^{ax} \, dx & v = \frac{1}{a} e^{ax} \end{array} \right.$$

$$\int e^{ax} \sin bx \, dx = \frac{1}{a} \sin bx \cdot e^{ax} - \frac{b}{a} \int e^{ax} \cos bx \, dx.$$

Como en el ejercicio anterior, sacamos:

$$\int e^{ax} \cos bx \, dx = \frac{e^{ax} (a \cos bx + b \sin bx)}{a^2 + b^2}$$

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax} (a \sin bx - b \cos bx)}{a^2 + b^2}$$

(Sturm, 355)

$$7. \int x \cdot \arcsen \frac{1}{2} \sqrt{\frac{2a-x}{a}} \, dx.$$

Sea

$$u = \arcsen \frac{1}{2} \sqrt{\frac{2a-x}{a}} = \arcsen z.$$

$$z = \frac{1}{2} \sqrt{\frac{2a-x}{a}}, z^2 = \frac{2a-x}{4a}, \sqrt{1-z^2} = \frac{1}{2} \sqrt{\frac{2a+x}{a}}.$$

$$du = d \left(\arcsen \frac{1}{2} \sqrt{\frac{2a-x}{a}} \right) = \frac{dz}{\sqrt{1-z^2}} = \frac{1}{2} \frac{1}{\sqrt{4a^2-x^2}} dx.$$

$$d v = x d x \quad \therefore \quad v = \frac{1}{2} x^2$$

$$\therefore \int u d v = \frac{1}{2} x^2 \operatorname{arc} \operatorname{sen} \frac{1}{2} \sqrt{\frac{2 a x - x}{a}} + \frac{1}{4} \int \frac{x^2}{\sqrt{4 a^2 - x^2}} d x$$

Integrando por partes la nueva integral, se obtiene:

$$y = \frac{1}{2} x^2 \operatorname{arc} \operatorname{sen} \sqrt{\frac{2 a x - x}{a}} + \frac{1}{2} a^2 \operatorname{arc} \operatorname{sen} \frac{x}{2 a} \\ - \frac{1}{8} x \sqrt{4 a^2 - x^2}$$

$$8. \int \frac{x^2 \operatorname{arc} \operatorname{tg} x}{1+x^2} d x = \int x^2 \operatorname{arc} \operatorname{tg} x d (\operatorname{arc} \operatorname{tg} x)$$

Sea

$$u = x^2 \quad \therefore \quad d u = 2 x d x$$

$$d v = \operatorname{arc} \operatorname{tg} x d (\operatorname{arc} \operatorname{tg} x) \quad \therefore \quad v = \frac{1}{2} \operatorname{arc}^2 \operatorname{tg} x$$

$$\int u d v = \frac{1}{2} x^2 \operatorname{arc}^2 \operatorname{tg} x - \int x \operatorname{arc}^2 \operatorname{tg} x d x.$$

Sea

$$u = x \cdot \operatorname{arc} \operatorname{tg} x \quad \therefore \quad d u = \frac{x d x}{1+x^2} + \operatorname{arc} \operatorname{tg} x d x$$

$$d v = \operatorname{arc} \operatorname{tg} x d x \quad \therefore \quad v = x \operatorname{arc} \operatorname{tg} x - L (1+x^2)^{\frac{1}{2}}$$

Sustituyendo i reduciendo, resulta:

$$y = \left(x - \frac{1}{2} \operatorname{arc} \operatorname{tg} x \right) \operatorname{arc} \operatorname{tg} x - \frac{1}{2} \sqrt{1+x^2}$$

(Grégory, 257)

$$9. \int \frac{e^{\operatorname{arc} \operatorname{tg} x}}{(1+x^2)^{\frac{3}{2}}} dx.$$

Sea

$$u = e^{\operatorname{arc} \operatorname{tg} x} \quad \dots \quad du = \frac{e^{\operatorname{arc} \operatorname{tg} x}}{1+x^2} dx$$

$$dv = \frac{dx}{\sqrt{(1+x^2)^3}} \quad \dots \quad v = \int \frac{x^{-3} dx}{(x^{-3} + 1)^{\frac{3}{2}}}$$

$$= \frac{x}{\sqrt{1+x^2}}$$

$$y = \frac{x e^{\operatorname{arc} \operatorname{tg} x}}{(1+x^2)^{\frac{3}{2}}} - \int \frac{x \cdot e^{\operatorname{arc} \operatorname{tg} x}}{(1+x^2)^{\frac{3}{2}}} dx$$

Sea

$$u = e^{\operatorname{arc} \operatorname{tg} x} \quad \dots \quad du = \frac{e^{\operatorname{arc} \operatorname{tg} x}}{1+x^2} dx$$

$$dv = \frac{x dx}{(1+x^2)^{\frac{3}{2}}} \quad \dots \quad v = -\frac{1}{\sqrt{1+x^2}}$$

$$y = \frac{x+1}{2\sqrt{1+x^2}} \operatorname{arctg} x$$

(Brahm, 35)

72. *Reducciones sucesivas.*—Es el caso jeneral de la doble integracion por partes. Se emplea este método en la integracion de la potencia n^a de una funcion, tal como:

$$d y = (f x)^n d x:$$

hacemos

$$u = (f x)^n \quad \therefore \quad d u = n (f x)^{n-1} f' (x) d x$$

$$d v = d x \quad \therefore \quad v = x;$$

sustituimos en la fórmula

$$\int u d v = u v - \int v d u$$

i resulta:

$$\int (f x)^n d x = n \int (f x)^{n-1} f' (x) d x.$$

De este modo se ha conseguido rebajar en una unidad la potencia. Haciendo $n-1$ integraciones por partes sucesivas, reducimos al fin la potencia a una forma elemental i conocida.

El mismo método se aplica a potencias de la forma

$$(f x)^n (\phi x)^m.$$

$$1. \int L^n x \, dx. \text{ Sea } \left| \begin{array}{ll} u = L^n x & \dots \, du = n L^{n-1} x \frac{dx}{x} \\ dv = dx & \dots \, v = x \end{array} \right.$$

$$\int L^n x \, dx = x L^n x - n \int L^{n-1} x \, dx$$

Es evidente que la nueva integral es de forma mas elemental que la primitiva.

En lugar de hacer con la nueva integral una segunda integracion por partes, basta suponer sucesivamente $n=1, 2, 3, \dots$:

$$n=1, \int L x \, dx = x (L x - 1)$$

$$n=2, \int L^2 x \, dx = x (L^2 x - 2 L x + 2)$$

$$n=3, \int L^3 x \, dx = x (L^3 x - 3 L^2 x + 6 L x - 6)$$

$$n=4, \int L^4 x \, dx = x (L^4 x - 4 L^3 x + 12 L^2 x - \dots)$$

en jeneral

$$\int L^n x \, dx = x (L^n x - n L^{n-1} x + n(n-1) L^{n-2} x - \dots)$$

(Brahya, 36)

$$2. \int \frac{1}{L x} \, dx. \text{ Sea } \left| \begin{array}{ll} u = L^{-1} x & \dots \, du = -L^{-2} x \frac{dx}{x} \\ dv = dx & \dots \, v = x \end{array} \right.$$

$$\int \frac{L x}{x} \, dx = x L^{-1} x + \int \frac{dx}{L^2 x}$$

$$\text{Sea } \left\{ \begin{array}{l} u = L^{-2} x \quad \dots \quad du = -2 L^{-3} x \frac{dx}{x} \\ dv = dx \quad \dots \quad v = x \end{array} \right.$$

$$\int \frac{dx}{L^2 x} = x L^{-2} + 2! \int L^{-3} x dx.$$

Procediendo de igual manera, se obtiene la serie:

$$\begin{aligned} \int \frac{1}{Lx} dx = & x (L^{-1} x + L^{-2} x + 2! L^{-3} x + \dots \\ & + n! L^{-(n+1)} x + \dots, \end{aligned}$$

(Peirce, 57)

$$3 \quad \int x^2 \cos x dx = \int x^2 d \sin x = x^2 \sin x - 2 \int x \sin x dx,$$

$$\begin{aligned} \int x \sin x dx &= - \int x d \cos x = -x \cos x + \int \cos x dx \\ &= -x \cos x + \sin x. \end{aligned}$$

Sustituyendo este valor en la primera igualdad, se tendrá:

$$\int x^2 \cos x dx = x^2 \sin x + 2 x \cos x - 2 \sin x.$$

(Sturm, 315)

$$4. \quad \int x^3 e^x dx = x^3 e^x - \int e^x \cdot 3 x^2 dx$$

$$= x^3 e^x - 3 x^2 e^x - \int e^x \cdot 6 x dx$$

$$= x^3 e^x - 3 x^2 e^x + 6 x e^x - 6 e^x$$

$$= e^x (x^3 - 3x^2 + 6x - 6)$$

(De Comberousse, 692)

$$5. \int \frac{dx}{(1+x^2)^n} \quad \text{Siendo} \quad dL v = \frac{dv}{v},$$

la fórmula

$$\int u dv = uv - \int v du$$

se puede escribir, como lo hizo Cauchy:

$$\int uv dL v = uv - \int uv dL u,$$

Suponiendo que u i v son proporcionales a las potencias x^2 i

$$\frac{1+x^2}{x^2};$$

tendremos:

$$dL \frac{1+x^2}{x^2} = 2 \left(\frac{x}{1+x^2} - \frac{1}{x} \right) dx = - \frac{2 dx}{x(1+x^2)}$$

$$\therefore \int \frac{dx}{(1+x^2)^n} = \int \frac{-x(1+x^2)^{-n+1}}{2} dL \frac{1+x^2}{x^2}$$

$$= \frac{x}{2(n-1)(1+x^2)^{n+1}} + \frac{2n-3}{2n-2} \int \frac{dx}{(1+x^2)^{n-1}}$$

(Moigno, II, 32)

73. *Fórmulas de reduccion.*—Los métodos de los números anteriores nos conducen naturalmente a fórmulas que, como la última, permiten integrar cualquiera potencia de una función. Tales son las fórmulas de reduccion, de las cuales daremos a conocer las mas importantes.

1. *Diferenciales binomias.*—Son expresiones de la forma

$$x^m (a + b x^n)^p dx$$

en la que m, n, p son números commensurables, i a, b constantes diferentes de cero.

Para integrar dicha diferencial hai que disminuir sus exponentes, i las fórmulas resultantes llámanse *de reduccion*.

1.º *Rebajar el exponente p.*

Cambiamos m en $m-1$ i sea $a + b x^n = X$; tendremos que integrar

$$y = \int x^{m-1} X^p dx.$$

$$\text{Sea } \left\{ \begin{array}{l} u = X^p \quad \dots \quad du = p X^{p-1} dx, \\ dv = x^{m-1} dx \quad \dots \quad v = \frac{x^m}{m} : \end{array} \right.$$

$$y = \frac{x^m X^p}{m} - \int \frac{x^m}{m} p X^{p-1} dx.$$

Aquí

$$dX = d(a + b x^n) = b n x^{n-1} dx,$$

$$y = \frac{x^m X^p}{m} - \frac{b n p}{m} \int x^{m+n-1} X^{p-1} dx.$$

Tal es la fórmula pedida.

2.º *Rebajar el exponente m.*—Traspongamos el término negativo de la fórmula anterior:

$$\int x^{m+n-1} X^{p-1} dx = \frac{x^m X^p}{b n p} - \frac{m}{b n p} \int x^{m-1} X^p dx$$

Cambiamos m en $m-n$, p en $p+1$:

$$\int x^{m-1} X^p dx = \frac{x^{m-n} X^{p+1}}{b n (p+1)} - \frac{m-n}{b n (p+1)} \int x^{m-n-1} X^{p+1} dx$$

3.º *Rebajar p i m .*—Puesto que

$$X^p = (a + b x^n)^p = X^{p-1} (a + b x^n),$$

tendremos que

$$\int x^{m-1} X^p dx = a \int x^{m-1} X^{p-1} dx + b \int x^{m+n-1} X^{p-1} dx$$

i en el caso 2.º anterior encontramos que:

$$\int x^{m+n-1} X^{p-1} dx = \frac{x^m X^p}{b n p} - \frac{m-n}{b n p} \int x^{m-1} X^p dx.$$

Reemplazamos este valor;

$$\begin{aligned} \int x^{m-1} X^p dx &= \frac{x^m X^p}{b n p} - \int \frac{n}{b n p} \int x^{m-1} X^p dx \\ &\quad + a \int x^{m-1} X^{p-1} dx. \end{aligned}$$

Como aquí aparece dos veces la integral primitiva, reducimos i despejamos:

$$\int x^{m-1} X^p dx = \frac{x^m X^p}{m+n p} + \frac{a n p}{m+n p} \int x^{m-1} X^{p-1} dx.$$

Estas diferenciales son integrables cuando sean enteros

$$p, q \text{ y } \frac{m+1}{n} - 1.$$

Daremos algunos ejemplos.

$$i. \int \frac{x^2 dx}{(a+bx^2)^{\frac{3}{2}}} = -\frac{x}{b\sqrt{a+bx^2}} + \frac{1}{b} \int \frac{dx}{\sqrt{a+bx^2}}$$

$$ii. \int \frac{x^m dx}{\sqrt{c^2-x^2}} = -\frac{x^{m-1}\sqrt{c^2-x^2}}{m} + \frac{m-1}{m} c^2 \int \frac{x^{m-2} dx}{\sqrt{c^2-x^2}}$$

$$iii. \int \frac{dx}{x^m \sqrt{a^2+x^2}} = -\frac{\sqrt{a^2+x^2}}{(m-1)a^2 x^{m-1}} - \frac{m-2}{(m-1)a^2} \int \frac{dx}{x^{m-1} \sqrt{a^2+x^2}}$$

(Todhunter, Int. Calc., 42/45)

En la integracion de las funciones se darán a conocer las diferentes formas que tienen las diferenciales binomias.

$$2. \int x^n \cdot e^{ax} dx. \quad \text{Sea} \quad \left| \begin{array}{l} u = x^n \quad \therefore du = nx^{n-1} dx \\ dv = e^{ax} dx \quad \therefore v = \frac{1}{a} e^{ax} \end{array} \right.$$

$$y = \frac{1}{a} x^n e^{ax} - \frac{n}{a} \int x^{n-1} e^{ax} dx$$

I. Sea $a=1$,

$$\int x^n e^x dx = x^n e^x - n \int x^{n-1} e^x dx$$

II. Sea $n=1$

$$\int x e^x dx = x e^x - e^x$$

III. $n=2$: $\int x^2 e^x dx = x^2 e^x - 2 \int x e^x dx$

$$= x^2 e^x - 2 (x e^x - e^x)$$

En jeneral:

IV. $\int x^n e^x dx = e^x [x^n - n x^{n-1} + n(n-1)x^{n-2} - \dots]$

V. Sea $n=4$:

$$\int e^{ax} x^4 dx = e^{ax} \left(\frac{x^4}{a} - \frac{4x^3}{a^2} + \frac{12x^2}{a^3} - \frac{24x}{a^4} + \frac{4!}{a^5} \right)$$

VI. Sea $n=5, a=-1$

$$\int \frac{x^5}{e} dx = -e^{-x} (x^5 + 5x^4 + 20x^3 + 60x^2 + 120x + 120)$$

VI. Sea n negativo:

$$\int \frac{e^{ax}}{x^n} dx = -\frac{e^{ax}}{(n-1)x^{n-1}} + \frac{a}{n-1} \int \frac{e^{ax}}{x^{n-1}} dx$$

VIII. Sea $x = -1$;

$$\int \frac{e^{ax}}{x} dx = Lx + ax + \frac{a^2 x^2}{1 \cdot 2^2} + \frac{a^3 x^3}{1 \cdot 2 \cdot 3^2} +$$

IX. Sea $n=3$, $a=1$

$$\int \frac{e^x}{x^3} dx = -\frac{e^x}{2x^2} (1+x) + \frac{1}{2} \int \frac{e^x}{x} dx$$

(Grégory, 276-277)

3. $x^m L^n x \cdot dx$. (Grégory, 277)

Sea

$$u = L^n x \quad \therefore du = n L^{n-1} x \frac{dx}{x}$$

$$dv = x^m dx \quad \therefore v = \frac{x^{m+1}}{m+1}$$

$$\int \frac{x^{m+1} L^n x}{m+1} - \frac{n}{m+1} \int x^m L^{n-1} x dx$$

I. Sea $n=1$,

$$\int x^m L x dx = \frac{x^{m+1} L x}{m+1} - \frac{1}{m+1} - \frac{x^{m+1}}{m+1}$$

II. Sea $n=2$,

$$\int x^m L^2 x \, dx = \frac{x^{m+1} L^2 x}{m+1} - \frac{2}{m+1} \cdot \\ - \frac{x^{m+1} L x}{m+1} + \frac{x^{m+1}}{(m+1)^2}$$

III. Sea $m=3, u=2$

$$\int x^3 L^2 x \, dx = \frac{1}{4} x^4 \left(\log^2 x - \frac{1}{2} L x + \frac{1}{8} \right)$$

IV. Sea $m=1, n=3$

$$\int x L^3 x \, dx = \frac{1}{2} x^2 \left(L^3 x - \frac{3}{2} L^2 x - \frac{3}{4} \right)$$

4. $\int \sin^m x \cos^n x \, dx$. Esta es una de las integrales que mas aplicaciones tiene.

I. Hacemos algebraica la derivada, poniendo

$$z = \sin x \quad \sqrt{1-z^2} = \cos x, \, dx = \frac{1}{\sqrt{1-z^2}} dz$$

$$\sin^m x \cos^n x \, dx = z^m (1-z^2)^{\frac{n-1}{2}} dz$$

II. Integracion por partes.

Sea

$$u = \cos^{n-1} x \quad dv = (n-1) \cos^{n-2} x \sin x \, dx$$

$$dv = \sin^m x \cos x \, dx \quad \therefore \quad v = \frac{\sin^{m+1} x}{m+1}$$

$$\therefore \int \sin^m x \cos^n x \, dx$$

$$= \frac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \frac{n-1}{m+1} \int \sin^{m+2} x \cos^{n-1} x \, dx.$$

Para disminuir el nuevo esponente de sen, empleamos la trasformacion que sigue:

$$\sin^{m+2} x = \sin^m x (1 - \cos^2 x):$$

$$\therefore \int \sin^m x \cos^n x \, dx$$

$$= \frac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \frac{n-1}{m+1} \int \sin^m x \cos^{n-1} x \, dx$$

$$- \frac{n-1}{m+1} \int \sin^m x \cos^n x \, dx.$$

Aquí aparece la integral primitiva; trasponemos i despejamos:

$$\int \sin^m x \cos^n x \, dx = \frac{\sin^{m+1} x \cos^{n-1} x}{m+1}$$

$$= \frac{n-1}{m+n} \int \sin^m x \cos^{n-2} x \, dx$$

III. Haciendo $u = \sin^{m-1} x$, $dv = \cos^n x \sin x \, dx$, se obtiene m rebajado en 2 unidades:

$$\int \operatorname{sen}^m x \cos^n x \, dx = - \frac{\operatorname{sen}^{m-1} x \cos^{n+1} x}{m+n}$$

$$= \frac{m-1}{m+n} \int \operatorname{sen}^{m-2} x \cos^n x \, dx$$

IV. Sea $n=0$, la segunda fórmula nos da:

$$\int \operatorname{sen}^m x \, dx = - \frac{\operatorname{sen}^{m-1} x \cos x}{m} + \frac{m-1}{m} \int \operatorname{sen}^{m-2} x \, dx.$$

V. Sea $m=0$, la primera nos da:

$$\int \cos^n x \, dx = \frac{\operatorname{sen} x \cos^{n-1} x}{n} + \frac{n-1}{n} \int \cos^{n-2} x \, dx$$

VI. Si m es par:

$$\begin{aligned} \int \operatorname{sen}^m x \, dx &= - \frac{\cos x}{m} \left[\operatorname{sen}^{m-1} x + \frac{m-1}{m-2} \operatorname{sen}^{m-3} x \right. \\ &+ \frac{(m-1)(m-3)}{(m-2)(m-4)} \operatorname{sen}^{m-5} x + \dots + \frac{(m-1) \dots 3 \cdot 1}{(m-2) \dots 4 \cdot 2} \operatorname{sen} x \left. \right] \\ &+ \frac{(m-1) \dots 3 \cdot 1}{(m-2) \dots 4 \cdot 2} \cdot \frac{x}{m} \end{aligned}$$

VII. m es negativo, $n=0$:

$$\int \frac{dx}{\operatorname{sen}^m x} = - \frac{\cos x}{(m-1) \operatorname{sen}^{m-1} x} + \frac{m-2}{m-1} \int \frac{dx}{\operatorname{sen}^{m-2} x}$$

VIII. n es negativo:

$$\int \frac{\operatorname{sen}^m x \, dx}{\cos^n x} = -\frac{\operatorname{sen}^{m-1} x}{(m-n) \cos^{n-1} x} + \frac{m-1}{m-n} \int \frac{\operatorname{sen}^{m-2} x \, dx}{\cos^n x}.$$

IX. Cambiando n por $-m$,

$$\int \operatorname{tg}^m x \, dx = \frac{1}{m-1} \operatorname{tg}^{m-1} x - \int \operatorname{tg}^{m-2} x \, dx$$

X. $m=2, n=0$:

$$\int \operatorname{sen}^2 x \, dx = -\frac{1}{2} (\operatorname{sen} x \cos x - x)$$

$$\text{XI. } \int \frac{dx}{\operatorname{sen}^3 x} = -\frac{\cos x}{2 \operatorname{sen}^2 x} + \frac{1}{2} L \operatorname{tg} \frac{x}{2}$$

$$\text{XII. } \int \frac{\operatorname{sen}^4 x}{\cos x} \, dx = -\frac{\operatorname{sen} x}{3} (\operatorname{sen}^2 x + 3) + L \operatorname{tg} \left(\frac{1}{4} \pi + \frac{1}{2} x \right)$$

$$\text{XIII. } \int \operatorname{sen}^3 x \cos^2 x \, dx = -\frac{1}{5} \cos^3 x \left(\operatorname{sen}^2 x + \frac{2}{3} \right)$$

$$\text{XIV. } \int \frac{dx}{\operatorname{sen} x \cos^4 x} = \frac{1}{3 \cos^3 x} + \frac{1}{\cos x} + L \operatorname{tg} \frac{1}{2} x$$

$$\text{XV. } \int \operatorname{tg}^4 x \, dx = \frac{1}{3} \operatorname{tg}^3 x - \operatorname{tg} x + x.$$

$$\begin{array}{l} 5. \int x^n \cos x \, dx. \quad \left| \begin{array}{l} u = x^n \\ dv = \cos x \, dx \end{array} \right. \quad \therefore \quad \left| \begin{array}{l} du = n x^{n-1} \, dx \\ v = \operatorname{sen} x \end{array} \right. \end{array}$$

$$y = x^n \sin x - n \int x^{n-1} \sin x \, dx$$

$$u = x^{n-1} \quad du = (n-1)x^{n-2} \, dx$$

$$dv = \sin x \, dx \quad v = -\cos x$$

$$y = x^n \sin x + n x^{n-1} \cos x - n(n-1) \int x^{n-2} \cos x \, dx$$

$$I. \quad \int x^2 \cos x \, dx = x^2 \sin x + 2x \cos x - 2 \sin x$$

$$II. \quad \int x^3 \cos x \, dx = x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6 \cos x$$

$$III. \quad \int x \sin x \, dx = -x \cos x + \sin x$$

$$IV. \quad \int x^4 \sin x \, dx = -x^4 \cos x + 4x^3 \sin x + 12x^2 \cos x$$

$$-24x \sin x - 24 \cos x$$

74. INTEGRALES PRINCIPALES.—El estudio de los métodos i ejercicios que preceden, habrá dado a conocer al lector la forma i naturaleza de la Integracion, operacion que consiste en trasformar la diferencial propuesta hasta reducirla a una de las fórmulas simples llamadas *fundamentales*.

En el número 75 damos las fórmulas que a nuestro juicio son verdaderamente fundamentales: en realidad no tienen ningun valor práctico sino abstracto i jeneral.

Las fórmulas usuales a que hai que reducir las diferenciales que se quiere integrar, con mas propiedad las llamamos *principales*, en atencion a la importancia que tienen, es decir, a la frecuencia con que se necesitan en la Integracion.

Antes de formar este grupo de fórmulas principales, hemos comparado i discutido cada una de las integrales fundamentales que recomiendan los autores cuyos nombres hemos citado en los ejercicios: algunos, como Roberts, dan hasta 45

de estas fórmulas; otros, como Jordan, reducen su número a 7. Creemos que las 12 fórmulas que siguen son suficientes i abarcan todos los casos que pueden presentarse en la Integracion:

INTEGRALES PRINCIPALES

$$\text{I. signo:} \quad \int \pm dx = \pm x$$

$$\text{II. coeficiente:} \quad \int a dx = \pm ax$$

$$\text{III. exponente:} \quad \int x^n dx = \frac{x^{n+1}}{n+1}$$

$$\text{IV. Logaritmo:} \quad \int \frac{1}{x} dx = Lx$$

$$\text{V. arcotanjente:} \quad \int \frac{1}{1+x^2} dx = \text{arc tg } x$$

$$\text{VI. Log. radical:} \quad \int \frac{1}{1-x^2} dx = L \sqrt{\frac{1+x}{1-x}}$$

$$\text{VII. Log. binomio:} \quad \int \frac{1}{\sqrt{1+x^2}} dx = L(x + \sqrt{1+x^2})$$

$$\text{VIII. arco seno:} \quad \int \frac{1}{\sqrt{1-x^2}} dx = \text{arc sen } x$$

$$\text{IX. esponencial:} \quad \int e^x dx = e^x$$

$$\text{X. coseno:} \quad \int \text{sen } x dx = -\cos x$$

$$\text{XI. seno:} \quad \int \cos x \, dx = \sin x$$

$$\text{XII. tanjente:} \quad \int \sec^2 x \, dx = \operatorname{tg} x$$

Para integrar las demas diferenciales, sean simples o compuestas, se reducen, como ya se dijo, a una de estas doce fórmulas, trasformándolas mediante uno de los métodos ya estudiados i que pueden quedar comprendidos en los cuatro siguientes:

$$\text{XIII. Transformaciones algebraicas:} \quad \int f'(x) \, dx = \int df \, x$$

$$\text{XIV. Descomposiciones:} \quad \int (du \pm dv) = \int du \pm \int dv$$

$$\text{XV. Sustitucion:} \quad \int f(x) \, dx = \int f(\phi z) \phi' z \, dz$$

$$\text{XVI. Integracion por partes:} \quad \int u \, dv = uv - \int v \, du.$$

A continuacion van las doce fórmulas anteriores seguidas de las diferentes derivadas que dan como fundamentales los autores citados:

$$\text{I.} \quad \int \pm dx.$$

$$\text{II.} \quad \int a \, dx.$$

$$\text{III.} \quad \int x^n dx: \quad x \pm n, \quad x \frac{1}{n}, \quad a x^n, \quad n x^{n-1}$$

$$\frac{1}{2\sqrt{x}}, \quad (x \pm a) \pm u, \quad (ax + b)$$

$$f(x)^n f'(x)$$

$$\text{IV. } \int \frac{1}{x} dx: \pm \frac{1}{x}, \frac{a}{x}, \frac{1}{x-a}, \frac{1}{ax+b}, \frac{f'(x)}{f(x)}$$

$$\text{V. } \int \frac{1}{1+x^2} dx: -\frac{1}{1+x^2}, \frac{1}{a^2+x^2}$$

$$\text{VI. } \int \frac{1}{1-x^2} dx: \pm \frac{1}{1-x^2}, \pm \frac{1}{a^2-x^2}, \pm \frac{1}{x^2-a^2}$$

$$\text{VII. } \int \frac{1}{\sqrt{1+x^2}} dx: \pm \frac{1}{\sqrt{x \pm a^2}}$$

$$\text{VIII. } \int \frac{1}{\sqrt{1-x^2}} dx: -\frac{1}{\sqrt{1-x^2}}, \pm \frac{1}{\sqrt{a^2-x^2}},$$

$$\frac{1}{\sqrt{2x-x^2}}, \pm \frac{dx}{x \sqrt{x^2-a^2}}, \frac{dx}{x \sqrt{a^2 \pm x^2}}$$

$$\text{IX. } \int e^x dx: e^{ax}, e^{ax+b}, a^x, a^{bx}, a^x L a$$

$$\text{X. } \int \sin x dx: \pm \sin ax.$$

$$\text{XI. } \int \cos x dx: \cos ax, \operatorname{tg} x, \cot x, \sec x, \operatorname{cosec} x.$$

$$\text{XII. } \int \sec^2 x dx: \pm \sec^2 ax$$

$$\operatorname{cosec}^2 x, \frac{\sin x}{\cos^2 x}, \frac{\cos x}{\sin^2 x}$$

$$\frac{1}{\sin x \cos x}$$

75. *Formas fundamentales.*—Las fórmulas principales anteriores se pueden reducir, a su vez, a otros mas simples, que consideramos como fundamentales; tales son las dos siguientes:

$$\int dx = x, \quad (A)$$

$$\int -\frac{1}{x} dx = L x. \quad (B)$$

La primera es la definicion misma del Cálculo Integral, i la segunda es el logaritmo natural de la funcion.

Se establecen estas dos fórmulas invirtiendo las reglas de la diferenciacion.

Las llamamos *formas fundamentales*, porque son las mas simples en cuanto a la forma, es decir, son irreducibles o no se pueden descomponer en otras mas sencillas: i porque con su ayuda i con los métodos de Integracion, se integran todas las diferenciales que hasta hoy se han podido integrar.

Conforme con esto, vamos a hacer la integracion de las diferenciales principales, lo que, aun cuando no es mas que un ejercicio de cálculo, servirá para completar los conocimientos anteriores.

I. $\int \pm dx.$

Una sencilla trasformacion nos da:

$$\int \pm dx = \int d(\pm x) = \pm x.$$

O, si se quiere, hacemos

$$\pm x = u \quad \therefore \quad \pm dx = du$$

i se tendrá

$$\int \pm dx = \int du = u = \pm x.$$

II. $\int a dx$.

Como es el caso anterior:

$$\int a dx = \int da x = ax$$

$$= \int du = u = ax$$

III. $\int x^n dx$.

Integramos por partes:

Sea

$$u = x^n \quad \therefore du = nx^{n-1} dx$$

si sea

$$dv = dx \quad \therefore v = x$$

$$\therefore \int x^n dx = x^n \cdot x - n \int x^{n-1} \cdot x dx$$

La nueva integral se traspone i se obtiene:

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

1. $\int x^{\pm n} dx$. Hacemos $\pm n = m$ i se vuelve al ejemplo anterior:

$$\int x^{\pm u} dx = \int x^m dx = \frac{x^{m+1}}{m+1} = \frac{x^{\pm n+1}}{\pm n+1}$$

$$2. \int x^{\frac{1}{n}} dx = \int x^m dx = \frac{x^{m+1}}{m+1} = \frac{x^{\frac{1}{n}+1}}{\frac{1}{n}+1}$$

$$3. \int a x^n dx = a \int x^n dx = \frac{a x^{n+1}}{n+1}$$

$$4. \int n x^{n-1} dx = n \int x^m dx = n \cdot \frac{x^n}{n} = x^n.$$

$$5. \int \frac{1}{2\sqrt{x}} dx = \frac{1}{2} \int x^{-\frac{1}{2}} = \sqrt{x}.$$

$$6. \int (x \pm a)^{\pm n} dx = \int (x \pm a)^{\pm n} d(x \pm a)$$

$$= \frac{(x \pm a)^{\pm n+1}}{\pm n+1}$$

$$7. \int (ax+b)^n dx = \frac{1}{a} \int (ax+b)^n d(ax+b)$$

$$= \frac{(ax+b)^{n+1}}{a(n+1)}$$

$$8. \int (fx)^n f'x dx = \int (fx)^n d(fx) = \frac{(fx)^{n+1}}{n+1}$$

$$IV. \int \frac{1}{x} dx$$

Forma fundamental:

$$\int \frac{1}{x} dx = L x.$$

$$9. \int \pm \frac{1}{x} dx = \pm \int \frac{1}{x} dx = \pm L x$$

$$10. \int \frac{a}{x} dx = a \int \frac{1}{x} dx = a L x = L(x^a)$$

$$11. \int \frac{1}{x-a} dx = \int \frac{d(x-a)}{x-a} = L(x-a)$$

$$12. \int \frac{1}{a x + b} dx = \frac{1}{a} \int \frac{d(a x + b)}{a x + b} = \frac{1}{a} L(a x + b)$$

$$13. \int \frac{f' x}{f x} dx = \int \frac{d(f x)}{f x} = L(f x).$$

$$V. \int \frac{1}{1+x^2} dx.$$

La sustitucion trigonométrica nos dá, haciendo

$$x = \operatorname{tg} \theta \quad \therefore \quad dx = \sec^2 \theta d\theta,$$

$$1+x^2 = \sec^2 \theta.$$

$$\int \frac{1}{1+x^2} dx = \int \frac{\sec^2 \theta d\theta}{\sec^2 \theta} = \int d\theta = \theta = \operatorname{arc} \operatorname{tg} x.$$

$$14. \int -\frac{1}{1+x^2} dx \text{ Sea } x = \cot \theta \therefore dx = -\operatorname{cosec}^2 \theta d\theta:$$

$$\int -\frac{1}{1+x^2} dx = \int d\theta = \theta = \operatorname{arc} \cot x.$$

$$15. \int \pm \frac{1}{a^2 + x^2} dx = \frac{1}{a} \int \frac{\pm d\frac{x}{a}}{1 + \left(\frac{x}{a}\right)^2} = \begin{cases} \frac{1}{a} \operatorname{arc} \operatorname{tg} \frac{x}{a} \\ \frac{1}{a} \operatorname{arc} \cot \frac{x}{a} \end{cases}$$

$$\text{VI. } \int \frac{1}{1-x^2} dx.$$

El método de las fracciones parciales nos conduce a la trasformación:

$$\int \frac{1}{1-x^2} dx = \int \frac{1}{2} \left(\frac{1}{1+x} + \frac{1}{1-x} \right) dx$$

$$= L \sqrt{\frac{1+x}{1-x}}$$

$$16. \int -\frac{1}{1-x^2} dx = -L \sqrt{\frac{1+x}{1-x}} = L \sqrt{\frac{a-x}{1+x}}$$

$$17. \int \pm \frac{dx}{a^2 - x^2} = \pm \frac{1}{a} \int \frac{d\frac{x}{a}}{1 - \left(\frac{x}{a}\right)^2}$$

$$= \pm \frac{1}{a} L \sqrt{\frac{1 + \frac{x}{a}}{1 - \frac{x}{a}}} = \pm \frac{1}{a} L \sqrt{\frac{a+x}{a-x}}$$

$$18. \int \pm \frac{1}{x^2 - a^2} dx = \mp \int \frac{dx}{a^2 - x^2} = \mp \frac{1}{a} L \sqrt{\frac{a+x}{a-x}}$$

$$VII. \int \frac{1}{\sqrt{1+x^2}} dx.$$

Sustitucion algebraica.

Sea

$$1+x^2 = z^2 \quad \therefore \quad \frac{dx}{z} = \frac{dz}{x}.$$

$$\int \frac{1}{\sqrt{1+x^2}} dx = \int \frac{dx}{z} = \int \frac{d(x+z)}{z+z} \\ = L(x + \sqrt{1+x^2})$$

$$19. \int \pm \frac{1}{\sqrt{x^2 \pm a^2}} = \pm \int \frac{dx}{z} = \pm L(x \pm \sqrt{x^2 \pm a^2})$$

$$VIII. \int \frac{1}{\sqrt{1-x^2}} dx.$$

Sustitucion trigonométrica:

Sea

$$x = \text{sen } t \quad \therefore \quad dx = \cos t \, dt:$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \int \frac{\cos t \, dt}{\cos t} = \int dt = t = \text{arc sen } x.$$

$$20. \int \frac{-1}{\sqrt{1-x^2}} dx. \text{ Sea } x = \cos t \therefore dx = -\operatorname{sen} t dt$$

$$\int \frac{-1}{\sqrt{1-x^2}} dx = \int dt = t = \arccos x.$$

$$21. \int \pm \frac{1}{\sqrt{a^2 - x^2}} dx = \int \frac{\pm d \frac{x}{a}}{\sqrt{1 - \left(\frac{x}{a}\right)^2}} = \begin{cases} \arcsen \frac{x}{a} \\ \arccos \frac{x}{a} \end{cases}$$

$$22. \int \frac{1}{\sqrt{2x - x^2}} dx = \int \frac{-d(1-x)}{\sqrt{1 - (1-x)^2}} \\ = \arccos(1-x)$$

$$23. \int \frac{\pm 1}{x \sqrt{x^2 - a^2}} dx = \int \frac{\pm dx}{x^2 \sqrt{1 - \left(\frac{a}{x}\right)^2}} \\ = \frac{1}{a} \int \frac{\mp d \frac{a}{x}}{\sqrt{1 - \left(\frac{a}{x}\right)^2}} = \begin{cases} \frac{1}{a} \arccos \frac{a}{x} \\ \frac{1}{a} \arcsen \frac{a}{x} \end{cases} \\ = \begin{cases} \frac{1}{a} \operatorname{arcsec} \frac{x}{a} \\ \frac{1}{a} \operatorname{arccosec} \frac{x}{a} \end{cases}$$

$$24. \int \frac{\pm 1}{x \sqrt{a^2 \pm x^2}} = \pm \frac{1}{a} \int \frac{d \frac{a}{x}}{\sqrt{\left(\frac{a}{x}\right)^2 \pm 1}}$$

$$= \mp \frac{1}{a} L \left(\frac{a}{x} + \sqrt{\left(\frac{a}{x}\right)^2 \pm 1} \right)$$

$$= \mp \frac{1}{a} L \left(\frac{a}{x} + \sqrt{\frac{a^2 \pm x^2}{x^2}} \right)$$

$$= \mp \frac{1}{a} L \left(\frac{a + \sqrt{a^2 \pm x^2}}{x} \right)$$

IX. $\int e^x dx$. Sea $e^x = z \therefore de^x = e^x dx = dz$

$$\int e^x dx = \int dz = z = e^x$$

25. $\int e^{nx} dx$ Sea $e^{nx} = z \therefore dx = \frac{dz}{n}$

$$\int e^{nx} dx = \frac{1}{n} \int e^z dz = \frac{1}{n} e^{nx}.$$

26. $\int e^{ax+b} dx$. Sea $e^{ax+b} = z \therefore dx = \frac{dz}{a}$

$$\int e^{ax+b} dx = \frac{1}{a} \int e^z dz = \frac{1}{a} e^{ax+b}.$$

27. $\int a^x dx = \int e^{x L a} dx = \frac{1}{L a} e^{x L a} = \frac{1}{L a} a^x.$

En esta trasformacion suponemos:

$$e^z = a^x \quad z = x L a$$

$$\therefore a^x = e^{x L a}$$

$$28. \int a^{nx} dx = \frac{1}{n} \int a^z dz = \frac{1}{n} \cdot \frac{a^{nx}}{L a}.$$

$$29. \int a^x L a dx = L a \frac{a^x}{L a} = a^x$$

$$X. \int \sin x dx.$$

Segun las ecuaciones de Euler, tendremos:

$$\begin{aligned} \int \sin x dx &= \int \frac{e^{ix} - e^{-ix}}{2i} dx \\ &= \frac{1}{2i} (\int e^{ix} - \int e^{-ix}) dx \\ &= \frac{1}{2i} \cdot \left(\frac{e^{ix}}{i} + \frac{e^{-ix}}{i} \right) \\ &= -\frac{e^{ix} + e^{-ix}}{2} = -\cos x \end{aligned}$$

$$30. \int \sin ax dx = \frac{1}{a} \int \sin ax d(ax)$$

$$= -\frac{1}{a} \cos ax.$$

$$XI. \int \cos x dx = \int \sin (90^\circ - x) dx$$

$$= \int -\sin(90^\circ - x) d(90^\circ - x)$$

$$= \cos(90^\circ - x) = \sin x$$

$$31. \int \cos ax \, dx = \frac{1}{a} \int \cos ax \, d ax = \frac{1}{a} \sin ax.$$

$$32. \int \operatorname{tg} x \, dx = \int \frac{\sin x \, dx}{\cos x} = - \int \frac{d \cos x}{\cos x} = -L \cos x$$

$$33. \int \cot x \, dx = \int \frac{d \sin x}{\sin x} = L \sin x$$

$$34. \int \sec x \, dx = \int \frac{dx}{\cos x} = \int \frac{\cos x \, dx}{\cos^2 x} = \int \frac{d \sin x}{1 - \sin^2 x}$$

$$= L \sqrt{\frac{1 + \sin x}{1 - \sin x}} = L \operatorname{tg} \frac{1}{2} \left(\frac{1}{2} \pi + x \right)$$

$$35. \int \operatorname{cosec} x \, dx = \int \frac{dx}{\sin x} = \int \frac{-d \cos x}{1 - \cos^2 x}$$

$$= -L \sqrt{\frac{1 + \cos x}{1 - \cos x}} = L \operatorname{tg} \frac{1}{2} x$$

$$\text{XII. } \int \sec^2 x \, dx.$$

$$\int \sec^2 x \, dx = \int \frac{1}{\cos^2 x} \, dx = \int \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \, dx$$

$$= \int \frac{\cos x \, d \sin x + \sin x \, d \cos x}{\cos^2 x}$$

$$= \int d \frac{\operatorname{sen} x}{\cos x} = \operatorname{tg} x$$

$$36. \int \sec^2 a x \, dx = \frac{1}{a} \int \sec^2 a x \, da x$$

$$= \frac{1}{a} \operatorname{tg} a x$$

$$37. \int \operatorname{cosec}^2 x \, dx = \int \sec^2 (90^\circ - x) \, dx$$

$$= -\operatorname{tg} (90^\circ - x) = -\cot x.$$

$$38. \int \frac{\operatorname{sen} x}{\cos^2 x} \, dx = -\int \cos^{-1} x \, d \cos x$$

$$= \cos^{-1} x = \sec x.$$

$$39. \int \frac{\cos x}{\operatorname{sen}^2 x} \, dx = \int \operatorname{sen}^{-2} x \, d \operatorname{sen} x$$

$$= -\operatorname{sen}^{-1} x = -\operatorname{cosec} x.$$

$$40. \int \frac{1}{\operatorname{sen} x \cos x} \, dx = \int \frac{\frac{dx}{\cos^2 x}}{\frac{\operatorname{sen} x}{\cos x}} = \int \frac{d \operatorname{tg} x}{\operatorname{tg} x}$$

$$= L \operatorname{tg} x.$$

Valparaiso, Febrero 23 de 1909.

CÁRLOS WARGNY.

AUTORES CONSULTADOS

Appell	Grégory	Niewenglowski
Bertrand	Haag	Osborne
Boucharlat	Hall	
Bossut	Halley	Pauly
Bowser	Hermite	Peacock
Brahy	Hind	Peck
Briot	Houël	Peirce
Byerly	Humbert	Perry
	Hutton	Potts
Carnot		Price
Catalan	Jarrett	Pruvost
Cauchy	Jordan	
Claudel		Raffv
Comberousse	Karnack	Roberts
Cournot		Rouse Ball
Cox	Lacroix	Russell
	Lardner	
Dariès	Laurent	Serret
Davies	Legendre	Smith
Duhamel	Leibnitz	Sonnet
	Longchamps	Stoffaes
Edwards		Stone
Euler	Mac Laurin	Sturm
	Millar	
Fourcy	Moigno	Tannery
Francoeur	Montferrier	Timmermans
Frenet	Morgan	Tisserand
Freycinet	Murphy	Todhunter
Greenhill	Newton	Williamson